39. A Cyclic Vector in the Tensor Product of Irreducible Representations of Compact Groups

By Nobuhiko TATSUUMA
Department of Mathematics, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1982)

1. Let G be a non-abelian connected compact Lie group and T a maximal torus in G with Lie algebras $\mathfrak{g}$, $\mathfrak{t}$ respectively. With respect to $\mathfrak{t}$, we introduce a lexicographic order on the set of roots of $\mathfrak{g}_c$ (the complexification of $\mathfrak{g}$). And we denote by X_k^+ $(k=1,2,\cdots,n)$ (resp. X_k^-) root vectors for all positive roots (resp. negative roots) in this order.

Any unitary representation U is canonically able to be extended to a representation $U(X)$ of $\mathfrak{g}_c$. When U is irreducible, we can define uniquely its highest weight μ as a linear form on $\sqrt{-1}\,\mathfrak{t}$. The highest (resp. the lowest) weight vector v in U is characterized up to constant as a vector satisfying $U(X_k^+)v=0$ (resp. $U(X_k^-)v=0$) for all k.

In [1] Theorem 3′, C. Fronsdal and T. Hirai proved the following

Theorem. *Let* $v_1\in E_1$ (resp. $v_2\in E_2$) *be the non-zero highest* (resp. lowest) *weight vector for irreducible representation* U_1 (resp. U_2) *of* G. *Then the vector* $v_1\otimes v_2$ *in* $E_1\otimes E_2$ *is a cyclic vector for the tensor product* $U_1\otimes U_2$.

The purpose of this paper is to give another proof of this theorem.

2. Proof of Theorem. Since G is compact, we can assume that U_1, U_2 are unitary. And it is enough to show that for any irreducible component U in $U_1\otimes U_2$ with representation space E in $E_1\otimes E_2$,

(1) the vector $v_1\otimes v_2$ is not orthogonal to E.

By weight vectors $v_j^\alpha\in E_j$ $(\alpha=1,2,\cdots,m_j)$ $(v_1=v_1^1, v_2=v_2^1)$, any v in E is expanded in a unique way as

(2) $$v=\sum_{\alpha,\beta} a(v, v_1^\alpha, v_2^\beta)v_1^\alpha\otimes v_2^\beta.$$

Especially the highest weight vector w in U is written as

(3) $$w=\sum_\alpha v_1^\alpha\otimes u^\alpha \qquad (u^\alpha\in E_2),$$

here

$$u^\alpha=\sum_\beta a(w, v_1^\alpha, v_2^\beta)v_2^\beta.$$

The vector w satisfies for any k,

(4) $$U(X_k^+)w=\sum_\alpha U_1(X_k^+)v_1^\alpha\otimes u^\alpha+\sum_\alpha v_1\otimes U_2(X_k^+)u^\alpha=0.$$

Let the weight μ_1^γ be the highest among the set $\{\mu_1^\alpha\,;\,u^\alpha\neq 0 \text{ in } (4)\}$. Since the vector $U_1(X_k^+)v_1^\gamma$ has the weight higher than μ_1^γ, it must vanish for any k. This means $v_1^\gamma=v_1$, and therefore

(5) $$M\equiv\{\mu_2^\beta\,;\,\exists v\in E \text{ such that } a(v,v_1,v_2^\beta)\neq 0\}\neq\emptyset.$$

Let μ_2^δ be the lowest in M, and $v_0\in E$ a vector for which $a(v_0,v_1,v_2^\delta)\neq 0$. In the expansion

(6) $$U(X_k^-)v_0=\sum_\beta a(v_0,v_1,v_2^\beta)(v_1\otimes U_2(X_k^-)v_2^\beta+U_1(X_k^-)v_1\otimes v_2^\beta)+\cdots,$$

if $U_2(X_k^-)v_2^\beta\neq 0$, it has the weight lower than μ_2^β. Because of the selection of μ_2^δ, $U(X_k^-)v_2^\delta=0$ for any k. That is, $v_2^\delta=v_2$, and

(7) $$\langle v_0, v_1\otimes v_2\rangle=a(v_0,v_1,v_2^\delta)\neq 0.$$

This proves (1) directly.

3. The fact that $U_1\otimes U_2$ is cyclic, is valid more generally.

Proposition. *For any finite dimensional irreducible unitary representations U_j $(j=1,2)$ of a locally compact group G, the tensor product $U_1\otimes U_2$ is cyclic.*

A proof of this proposition is deduced from following Lemma 1 and wellknown Lemma 2.

Lemma 1. *Let U_j $(j=1,2,3)$ be finite dimensional irreducible unitary representations of locally compact group G. Then*

(8) $$[U_1\otimes U_2 : U_3]\leqq \operatorname*{Min}_{j=1,2,3}(\dim U_j).$$

Here $[D:U]$ is the multiplicity of U in D.

Proof. At first the following is trivial.

(9) $$[U_1\otimes U_2 : U_3]=(\dim U_1)(\dim U_2)/(\dim U_3).$$

Denote U^* the conjugation of U in the sense of G. W. Mackey [2], and $\mathbf{1}$ the unit representation of G. The standard theory of tensor product of finite dimensional unitary representations leads us to

(10) $$[U_1\otimes U_2 : U_3]=[U_1\otimes U_2\otimes U_3^* : \mathbf{1}]$$
$$=[U_2\otimes U_3^* : U_1^*]=[U_1\otimes U_3^* : U_2^*].$$

Combining (9) and (10), we get the result.

Lemma 2. *Finite dimensional unitary representation D of a locally compact group G is cyclic, if and only if*

(11) $$[D:U]\leqq \dim U,$$

for any irreducible unitary representation U.

References

[1] C. Fronsdal and T. Hirai: A remark on the tensor product of irreducible representations of a compact Lie group. Jap. J. Math., New series, **7**, 201–211 (1981).

[2] G. W. Mackey: Induced representations of locally compact groups I. Ann. of Math., **55**, 101–139 (1952).