20. *Spatial Growth of Solutions of a Non-Linear Equation*

By Kōhei UCHIYAMA
Nara Women's University

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1. Given a continuous function $M(u, \bar{u})$ of $(u, \bar{u}) \in [0, 1]^2$ and a nondecreasing function $F(x)$ on $\boldsymbol{R}=(-\infty, +\infty)$ with $\lim_{x\to-\infty} F(x)=0$, and $\lim_{x\to+\infty} F(x)=1$, let us consider the following evolution equation

$$(1) \qquad \frac{\partial u}{\partial t}=M(u, \bar{u}) \quad (u=u(x, t),\ x \in \boldsymbol{R},\ t>0)$$

where

$$\bar{u}=\bar{u}(x, t)=\int_{-\infty}^{+\infty} u(x-y, t)dF(y).$$

It is assumed throughout the paper that M has continuous partial derivatives $M_u=\partial M/\partial u$ and $M_{\bar{u}}=\partial M/\partial \bar{u}$, and satisfies

$$(2) \qquad \alpha\equiv M_{\bar{u}}(0, 0)>0, \qquad \beta\equiv M_u(0, 0)>-\alpha$$

$$(3) \qquad M(0, 0)=M(1, 1)=0;\ M_{\bar{u}}(u, \bar{u})\geqq 0 \text{ for } (u, \bar{u}) \in [0, 1]^2$$

$$(4) \qquad M(u, u)>0 \qquad \text{for } 0<u<1,$$

and that F is right-continuous and satisfies

$$(5) \qquad 0<F(0-)\leqq F(0)<1$$

and its bilateral Laplace transform

$$\psi(\theta)\equiv\int_{-\infty}^{+\infty} e^{\theta x}dF(x)$$

is convergent in a neighborhood of zero.

It is routine to see from (3) that for any Borel measurable function $f(x)$ taking values in [0, 1], there is a unique solution of (1), with initial condition $u(x, 0)=f(x)$, which is also confined in [0, 1] (we will consider only such solutions), and that if two initial functions satisfy $0\leqq f_1\leqq f_2\leqq 1$, the corresponding solutions preserve the inequality.

A typical example of M is $M(u, \bar{u})=\alpha\bar{u}-(\alpha+\beta)u\bar{u}+\beta u$. If we let $\beta=0$ in this example, (1) becomes the equation of simple epidemics (cf. [5])

$$(6) \qquad \frac{\partial u}{\partial t}=\alpha\bar{u}(1-u).$$

Another typical case is $M=\alpha(\bar{u}-u)+g(u)$, where g is continuously differentiable function with $g(0)=g(1)=0$, $g'(0)>0$ and $g(u)>0$ for $0<u<1$. If we replace, in this case, the compound Poisson operator $u \mapsto \bar{u}$ by the diffusion operator $u \mapsto \partial^2 u/\partial x^2$, a nonlinear diffusion equation

$$(7)\qquad \frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}+g(u)$$

appears. Concerning the equation (7) there are a number of works and it is shown among others that any solution of (7) with finite initial function propagates to the right and left with the asymptotic speed $2\sqrt{\alpha g'(0)}$, provided further $g(u)\leqq g'(0)u$ $(0\leqq u\leqq 1)$ (cf. [1]). The purpose of this note is to obtain an analogue for the equation (1). In a special case of (6) the result is obtained in [6] by an entirely different method (cf. also [2], [3] and [5]).

2. If $dF(x)$ is supported by a lattice containing zero, we denote by X the smallest one among such lattices; otherwise let $X=\boldsymbol{R}$. Set

$$c^*=\inf_{\theta>0}\frac{\alpha\psi(\theta)+\beta}{\theta}\quad\text{and}\quad c_*=-\inf_{\theta<0}\frac{\alpha\psi(\theta)+\beta}{|\theta|}.$$

The result of this note is

Theorem. *If the initial function is continuous and positive at least at one point of X and if $c_*<c_2<c_1<c^*$, then*

$$(8)\qquad \lim_{t\to\infty}\inf_{\substack{c_2t<x<c_1t\\ x\in X}} u(x,t)=1.$$

Remark. If it is further assumed that

$$(9)\qquad M(u,\bar{u})\leqq\alpha\bar{u}+\beta u\quad\text{for}\quad (u,\bar{u})\in[0,1]^2,$$

solutions of (1) *with* $u(x,0)=0$ *for* $x>0$ *propagate to the right with asymptotic speed* c^* in the sense of (8) and of the following

$$(10)\qquad \lim_{t\to\infty}\sup_{x>ct} u(x,t)=0\qquad\text{for } c>c^*.$$

(c^* may be negative; in such a case we should say that solutions recede to the left.) The relation (10) is easily seen by comparing solutions of (1) with those of the linear equation $\partial u/\partial t=\alpha\bar{u}+\beta u$ (cf. [3]). When the condition (9) is violated, the asymptotic speed for (1) could be larger than c^*, as is suggested from the diffusion case (7). Arguments for c_* are parallel.

3. For the proof of Theorem we prepare two lemmas.

Lemma 1. *Let $c_*<c_2<c_1<c^*$. Then there is a positive number δ such that if $0<\varepsilon$, $\lambda<\delta$, and $c_2\leqq c\leqq c_1$, the function*

$$(11)\qquad w(x)=\varepsilon\exp(-\lambda x^2)$$

is a c-substationary solution for (1), i.e.

$$(12)\qquad cw'+M(w,\bar{w})\geqq 0\qquad\left(w'=\frac{dw}{dx}\right).$$

Proof. Let $c_*<c<c^*$ and w be defined by (11). Then for small enough ε

$$M(w,\bar{w})(x)\geqq\{\beta+s(\varepsilon)+(\alpha+s(\varepsilon))\int\exp(2\lambda xy-\lambda y^2)dF(y)\}w(x)$$

where $s(\varepsilon)$ is a function of ε only and tends to zero as $\varepsilon\downarrow 0$. If we set

$$J(\theta, \lambda, \varepsilon)=\frac{1}{|\theta|}\{\beta+s(\varepsilon)+(\alpha+s(\varepsilon))\int \exp(\theta y-\lambda y^2)dF(y)\} \quad (\theta\neq 0),$$

then

(13) $\quad cw'(x)+M(w, \bar{w})(x)\geqq 2\lambda|x|\{-c \operatorname{sign} x+J(2\lambda x, \lambda, \varepsilon)\}w(x),$

where $|x| \operatorname{sign} x=x$. It is not difficult to see that $\underline{\lim}_{\varepsilon,\lambda\downarrow 0}\min_{\theta>0}J(\theta, \lambda, \varepsilon)\geqq c^*$. Now let $c_*<c_2<c_1<c^*$. Then we can choose $\delta_1>0$ so that if $0<\varepsilon$, $\lambda<\delta_1$, then $-c+J(2\lambda x, \lambda, \varepsilon)\geqq 0$ for $x>0$ and $c\leqq c_1$. Similarly if $0<\varepsilon$, $\lambda<\delta_2$, then $c+J(2\lambda x, \lambda, \varepsilon)\geqq 0$ for $x<0$ and $c\geqq c_2$. Thus the assertion of Lemma 1 follows from (13) by setting $\delta=\min(\delta_1, \delta_2)$.

Lemma 2. *Let $b_0=\sup\{x: F(x)<1\}$ $(0<b_0\leqq\infty)$ and ε be any positive number. Then the solution of $\partial v/\partial t=\varepsilon\bar{v}$ $(t>0)$ satisfies*

$$\lim_{x\to\infty, x\in X}\frac{1}{x\log x}\log(1/v(x, t))=\frac{1}{b_0} \qquad \textit{for } t>0,$$

provided that $v(x, 0)$ is nonnegative, bounded and continuous, and vanishes for $x>0$ but does not for some point of X.

Proof. Let $f(x)=v(x, 0)$ satisfy what is provided in the lemma and set

$$G_t(x)=\sum_{n=0}^{\infty}\frac{(\varepsilon t)^n}{n!}F^{*n}(x) \quad (t>0),$$

where F^{*n} denotes the n-fold convolution of F. Then

$$v(x, t)=\int f(x-y)dG_t(y).$$

Noting $F^{*n}(y)=1$ for $y>nb_0$, we see

$$v(x, t)\leqq(\sup_y f(y))\sum_{n=[x/b_0]}^{\infty}(\varepsilon t)^n/n!$$

and hence, by $\lim\log(n!)/n\log n=1$, $\underline{\lim}(1/x\log x)\log(1/v(x, t))\geqq 1/b_0$. To prove the opposite inequality

(14) $$\overline{\lim}\frac{1}{x\log x}\log(1/v(x, t))\leqq\frac{1}{b_0},$$

we can assume $f=I_{[0,h)}$ (the indicator function of $[0, h)$) for a positive h. Take $b<b_0$ arbitrarily and observe that for each n

$$v(x, t)\geqq\frac{(\varepsilon t)^n}{n!}\int_{nb+}^{\infty}I_{[0,h)}(x-y)dF^{*n}(y)=\frac{(\varepsilon t(1-F(b)))^n}{n!}(\tilde{F}^{*n}(x)-\tilde{F}^{*n}(x-h))$$

where $\tilde{F}(x)=(F(x\vee b)-F(b))/(1-F(b))$. Let

$$\mu=\int x d\tilde{F}(x) \quad \text{and} \quad \sigma=\left(\int(x-\mu)^2 d\tilde{F}(x)\right)^{1/2}.$$

First we assume that $b_0=\infty$ or $F(b_0)-F(b_0-)=0$ and that F is nonlattice. Then $\sigma>0$ and a central limit approximation (cf. [4, § 42]) implies

$$\tilde{F}^{*n}(x)-\tilde{F}^{*n}(x-h)=\frac{h}{\sigma\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right) \qquad \text{as } n=[x/\mu]\to\infty,$$

and so

$$\overline{\lim}\frac{1}{x\log x}\log(1/v(x,t))\leqq\lim\frac{1}{x\log x}\log([x/\mu]!)=\frac{1}{\mu}.$$

Since $b<\mu$ and b can be arbitrarily close to b_0, we obtain (14).

When F is lattice or $F(b_0)-F(b_0-)>0$, (14) is verified in a similar way too, if we note $v(x,t)=\int v(x-y,t/2)dG_{t/2}(y)$ after seeing $\inf_{|x|<L,\,x\in X}v(x,t/2)>0$ for any $L>0$. This inequality follows from the fact that the support of $dG_t(x)$ agrees with X, which would be easily seen in the case that F is centered lattice or F is not lattice. In the remaining case there are real numbers $0<\xi<d$ such that ξ/d is irrational and F has positive jumps at ξ and at $\xi-d$, and therefore $dG_t(x)$ has positive mass at each point of $H=\{n\xi+m(\xi-d): n,m=1,2,\cdots\}$. It is left to the reader to show that H is dense in $\boldsymbol{R}$. The proof of Lemma 2 is finished.

4. Proof of Theorem. Let $c_*<c_2'<c_2<c_1<c_1'<c^*$ and $u(x,t)$ be a solution of (1) with an initial function satisfying the condition of Theorem. After some comparison arguments, a crude application of Lemma 2 shows that for any $\lambda>0$, there is $\varepsilon>0$ such that $u(x,1)\geqq\varepsilon\exp(-\lambda x^2)$, $x\in X$. Accordingly it follows from Lemma 1 that there is a smooth positive function w such that $u(x,1)\geqq w(x)$, $x\in X$ and w satisfies (12) simultaneously for $c_2'\leqq c\leqq c_1'$. Let $u_*(x,t)$ be the solution of (1) starting from this w. Then

(15) $\qquad u_*(x,t)\leqq u(x,t+1)$ for all $x\in X$ and $t>0$.

Since $v(x,t)\equiv(\partial/\partial t)u_*(x+ct,t)$ satisfies $\partial v/\partial t=c(\partial v/\partial x)+A\overline{v}+Bv$, where A and B are bounded continuous functions of (x,t), and $v(\cdot,0)=cw'+M(w,\overline{w})$, we see $u_*(x+ct,t)$ is nondecreasing in t if $c_2'\leqq c\leqq c_1'$. Now let $c_2\leqq c\leqq c_1$ and $w_c(x)\equiv\lim_{t\to\infty}u_*(x+ct,t)$. Then w_c is a stationary solution of $\partial u/\partial t=c(\partial u/\partial x)+M(u,\overline{u})$. In other words, $w_c(x-ct)$ is a solution of (1). Since $w_c(x)\geqq u_*(x,0)=w(x)$, it is found that $w_c(x-ct)\geqq u_*(x,t)$, or, what is the same, $w_c(x+(c_1'-c)t)\geqq u_*(x+c_1't,t)$. This implies $\underline{\lim}_{x\to\infty}w_c(x)>0$. Similarly $\underline{\lim}_{x\to-\infty}w_c(x)>0$. Thus $\delta\equiv\inf_x w_c(x)>0$ and so we have $w_c(x-ct)\geqq y(t)$, where y is a solution of $dy/dt=M(y,y)$ with $y(0)=\delta$. By virtue of (4), $y(t)\uparrow 1$. Hence $w_c\equiv 1$. Consequently $u_*(ct,t)\uparrow 1$ for $c_1\leqq c\leqq c_2$. By (15) this completes the proof.

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