19. *The Stokes Operator in L_r Spaces*

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Introduction. In this paper we shall report that the Stokes operator generates a bounded analytic semigroup of class C_0 in L_r spaces. Moreover, we shall decide domains of fractional powers of the Stokes operator. To show these we shall construct the resolvent of the Stokes operator, using pseudodifferential operators.

Let D be a bounded domain in $\boldsymbol{R}^n$ with the smooth boundary S. Let $1<r<\infty$ and let X_r be the closure in $(L_r(D))^n$ of all smooth solenoidal vector fields with compact supports in D. Then there exists the continuous projection P_r from $\boldsymbol{L}_r(D)=(L_r(D))^n$ onto X_r; see Fujiwara-Morimoto [5]. We denote by $W_r^m(D)$ the Sobolev space of order m. Set $\boldsymbol{W}_r^m(D)=(W_r^m(D))^n$. Then we define the Stokes operator by $A_r=-P_r\Delta$ ($\Delta=\partial_{x_1}^2+\cdots+\partial_{x_n}^2$) whose domain is

$$D(A_r)=\{w\in \boldsymbol{W}_r^2(D)\cap X_r : w|_S=0\}.$$

Let $\varepsilon>0$, $\omega\geqq 0$ and let $\Sigma_{\varepsilon,\omega}$ denote the set of $\lambda\in \boldsymbol{C}$ such that $|\arg\lambda|\leqq\pi-\varepsilon$, $|\lambda|>\omega$. Then we have

Theorem 1. *For any $\varepsilon>0$ there exists a constant $C_{\varepsilon,r}$ independent of $f\in X_r$ and of $\lambda\in\Sigma_{\varepsilon,0}$ such that*

$$\|(\lambda+A_r)^{-1}f\|\leqq C_{\varepsilon,r}|\lambda|^{-1}\|f\|, \tag{1}$$

where $\|\ \|$ denotes the norm of $\boldsymbol{L}_r(D)$. Consequently, $-A_r$ generates a bounded analytic semigroup of class C_0 in X_r.

Remark. This result is partially known by Solonnikov [14]; he proved (1) for $|\arg\lambda|\leqq\delta+\pi/2$, where $\delta\geqq 0$ is small. Our result is new in the following two points:

i) We prove that the estimate (1) holds for larger domain of λ, that is, $\lambda\in\Sigma_{\varepsilon,0}$ for *any* positive ε.

ii) We construct the resolvent $(\lambda+A_r)^{-1}$ explicitly. This enables us to describe the domain of fractional power A_r^α of A_r. For the case of the Laplace operator the corresponding result is well known; see Fujiwara [4] and Seeley [12].

By Theorem 1 we can define A_r^α. Concerning A_r^α we have

Theorem 2. *For any $\varepsilon>0$ there exists a constant $M_{\varepsilon,r}$ independent of $f\in X_r$, $-1\leqq a<0$, $b\in\boldsymbol{R}$ such that*

$$\|A_r^{a+ib}f\|\leqq M_{\varepsilon,r}e^{\varepsilon|b|}\|f\|,\quad (i=\sqrt{-1}).$$

This implies, like Kato [6],

$$D(A_r^\alpha)=[X_r, D(A_r)]_\alpha,\qquad 0<\alpha<1, \tag{2}$$

where [,] denotes the complex interpolation of two Banach spaces. Let $B_r = -\Delta$ with $D(B_r)=\{w \in W_r^2(D): w|_S=0\}$. Then, by (2) we have

Theorem 3. $D(A_r^\alpha)=X_r \cap D(B_r^\alpha)$, $0<\alpha<1$.

This generalizes the result in Fujita-Morimoto [3]; they proved Theorem 3 when $r=2$.

Theorems 1 and 3 are useful in treating the initial value problem for the Navier-Stokes equations; see Sobolevskii [13], Fujita-Kato [2], Kato-Fujita [7], Solonnikov [14], Miyakawa [10].

§ 1. The resolvent of the Stokes operator. To show Theorems 1 and 2 it is essential to construct the resolvent $(\lambda+A_r)^{-1}$. We can transform the equation $(\lambda+A_r)u=f$ in X_r into the following Stokes equations:

$$\text{(S)}\qquad \begin{aligned} (\lambda-\Delta)u+\nabla p&=f \quad \text{in} \quad D,\\ \operatorname{div} u&=0 \quad \text{in} \quad D,\\ u|_S&=0 \quad \text{on} \quad S, \end{aligned}$$

where p is some scalar function. Since f determines u, we denote u by $u=G_\lambda f$. When $\lambda=0$, Odqvist [11] constructed the kernel functions of G_0; see, for the details, Ladyzhenskaya [9] and the papers cited there.

To construct the resolvent G_λ we use the potential theoretical discussions. Set

$$(3)\qquad k_\lambda^{ij}(\xi)=(\delta^{ij}-\xi_i\xi_j/|\xi|^2)/(\lambda+|\xi|^2), \quad \xi \in \boldsymbol{R}^n, \quad 1\leqq i,\ j\leqq n,$$

where δ^{ij} denotes Kronecker's delta and $|\xi|^2=\xi_1^2+\cdots+\xi_n^2$. Identify f with its extension to $\boldsymbol{R}^n$ which vanishes outside D. Then we define the hydrodynamic potential of f by

$$(K_\lambda f)(x)=(\mathcal{F}^{-1}k_\lambda \mathcal{F} f)(x),$$

where $\mathcal{F}$ denotes the Fourier transformation with respect to x. We can easily see that the volume potential $u'=K_\lambda f$ satisfies the equations

$$\text{(K)}\qquad \begin{aligned} (\lambda-\Delta)u'+\nabla p'&=f \quad \text{in} \quad \boldsymbol{R}^n,\\ \operatorname{div} u'&=0 \quad \text{in} \quad \boldsymbol{R}^n, \end{aligned}$$

where p' is some scalar function on $\boldsymbol{R}^n$. Let $z=N\varphi$ satisfy

$$\text{(N)}\qquad \begin{aligned} &\Delta z=0 \quad \text{in} \quad D,\\ &\left.\frac{\partial z}{\partial \nu}\right|_S=\varphi \quad \text{on} \quad S, \qquad \int_D z(x)dx=0, \end{aligned}$$

where ν_x denotes the unit interior normal vector to S at $x \in S$. Let $\langle\ ,\ \rangle$ be the standard inner product in $\boldsymbol{R}^n$. Let $W_{r,\tau}^s(S)$ be the set of $g \in W_r^s(S)$ satisfying $\langle g, \nu\rangle=0$. Let $v=V_\lambda g$ be the solution of the equations

$$\text{(T)}\qquad \begin{aligned} (\lambda-\Delta)v+\nabla q&=0 \quad \text{in} \quad D,\\ \operatorname{div} v&=0 \quad \text{in} \quad D,\\ v|_S&=g \in W_{r,\tau}^0(S) \quad \text{on} \quad S,\\ \langle v|_S, \nu\rangle&=0 \quad \text{on} \quad S, \end{aligned}$$

where q is some scalar function; we call this problem (T) the Dirichlet problem with tangential data. Set

$$M_\lambda f=\gamma_S K_\lambda f-\gamma_S \nabla N\langle \nu, \gamma_S K_\lambda f\rangle,$$

where $\gamma_S w=w|_S$. Then, by the definition of N we have $\langle M_\lambda f, \nu\rangle=0$. By (S), (K), (N), (T) we can easily prove

Proposition 1. $G_\lambda f=K_\lambda f-\nabla N\langle \nu, \gamma_S K_\lambda f\rangle-V_\lambda M_\lambda f$.

This procedure is found in Fabes-Lewis-Riviere [1]. Our next problem is to constract V_λ.

§ 2. Pseudodifferential operators. In order to construct V_λ we introduce a symbol class of pseudodifferential operators with a parameter λ.

Definition. Let m and k be real numbers. Then we denote by $S^{m;k}(\boldsymbol{R}^n)$ the set of all $p_\lambda \in C^\infty(\boldsymbol{R}^n\times \boldsymbol{R}^n)$ $(\lambda \in \boldsymbol{C}\backslash(-\infty, 0])$ such that for all multi-indices α, β and positive numbers ε, ω

$$C_{\alpha,\beta,\varepsilon,\omega}=\sup\{\langle\xi\rangle^{|\alpha|-m}\langle\lambda;\xi\rangle^{-k}|\partial_\xi^\alpha \partial_x^\beta p_\lambda(x, \xi)|:(x, \xi)\in \boldsymbol{R}^n\times \boldsymbol{R}^n,\ \lambda\in \Sigma_{\varepsilon,\omega}\}$$

is finite; here $\langle\lambda;\xi\rangle$ denotes $(|\lambda|+|\xi|^2+1)^{1/2}$ and $\langle\xi\rangle=\langle 0;\xi\rangle$.

Example. Let $k_\lambda(\xi)$ be as in (3). Let $\varphi(s)\in C^\infty(\{s\geqq 0\})$ satisfy

$$\varphi(s)=\begin{cases} 0: 0\leqq s\leqq 1 \\ 1: 2\leqq s \end{cases}.$$

Set $\psi(\xi)=\varphi(|\xi|)$. Then we get $\psi k_\lambda(\xi)\in S^{0;-2}(\boldsymbol{R}^n)$.

When a linear operator P_λ: $\mathcal{S}\to\mathcal{S}$ has the expression

$$(P_\lambda w)(x)=\frac{1}{(2\pi)^n}\int e^{ix\xi}p_\lambda(x, \xi)(\mathcal{F}w)(\xi)d\xi, \quad w\in\mathcal{S}$$

with $p_\lambda(x, \xi)\in S^{m;k}(\boldsymbol{R}^n)$, we call P_λ a pseudodifferential operator with its total symbol $\sigma(P_\lambda)=p_\lambda(x, \xi)$. Let Q_λ be another pseudodifferential operator with the total symbol $q_\lambda\in S^{m';k'}(\boldsymbol{R}^n)$. Then, like usual theory of pseudodifferential operators, $P_\lambda Q_\lambda$ is again a pseudodifferential operator with its total symbol $\sigma(P_\lambda Q_\lambda)\in S^{m+m';k+k'}(\boldsymbol{R}^n)$. However, to separate the part of the highest order in $\sigma(P_\lambda Q_\lambda)$ with respect to λ we need additional assumptions on p_λ.

Proposition 2. *Suppose that $\partial_{\xi_j}p_\lambda$ belongs to $S^{m;k''}(\boldsymbol{R}^n)(k''<k)$ for all j $(1\leqq j\leqq n)$. Then we have*

$$\sigma(P_\lambda Q_\lambda)=p_\lambda q_\lambda+r_\lambda \quad \textit{with} \quad r_\lambda\in S^{m+m';k'+k''}(\boldsymbol{R}^n).$$

§ 3. The Dirichlet problem with tangential data. Let $\{U_\kappa\}$ be a finite open covering of S which consists of local coordinates neighbourhoods of S; we denote by $F_{\kappa,0}$ the diffeomorphism from the closed unit ball $B=\{(z', 0)\in \boldsymbol{R}^{n-1}\times \boldsymbol{R}: |z'|^2\leqq 1\}$ onto $\overline{U}_\kappa$. Let $\{\chi_\kappa\}$ be a partition of unity subordinate to $\{U_\kappa\}$. Let Y_λ^κ be an $n\times n$ matrix of pseudodifferential operators on $\boldsymbol{R}^{n-1}$ satisfying to conditions:

(Y1) $\sigma(Y_\lambda^\kappa)(z', \zeta')=y_\lambda^\kappa(z', \zeta')\in S^{0;1}(\boldsymbol{R}^{n-1})$,

(Y2) y_λ^κ vanishes outside a compact set in B.

Define the operator $F_{\kappa,0}^*$ by $(F_{\kappa,0}^* f)(z')=f(F_{\kappa,0}(z'))$, for any $f\in C_0^\infty(U_\kappa)$.

Then we set $Y_\lambda^{\kappa *}=F_{\kappa,0}^{-1*}Y_\lambda^\kappa F_{\kappa,0}^*$ and

$$(Y_\lambda w)(x)=\sum_\kappa (Y_\lambda^{\kappa *}\chi_\kappa w)(x), \qquad w\in(\mathscr{D}'(S))^n.$$

We say that a bounded linear transformation P_λ in $W_r^s(S)$ (for all $s\in \boldsymbol{R}$) belongs to class $S(k)$ if the estimate

$$|P_\lambda w|_{W_r^s(S)}\leqq K_{s,\varepsilon,\omega}|\lambda|^{(k-d)/2}|w|_{W_r^{s+d}(S)},\ \ w\in W_r^{s+d}(S),\ \lambda\in\Sigma_{\varepsilon,\omega},\ k\leqq d\leqq 0$$

is valid for some constant $K_{s,\varepsilon,\omega}$. Roughly speaking, the class $S(k)$ contains the space of pseudodifferential operators whose total symbols belong to $S^{0;k}$; recall L_r boundedness theorem of pseudodifferential operators (cf. Kumano-go and Nagase [8]).

To construct $V_\lambda g$ we compute

$$T_\lambda=\gamma_S K_\lambda(\delta_S\otimes Y_\lambda\cdot)$$

modulo $S(-1)$. Let F_κ be a mapping defined on $B_\mu=B\times[-\mu,\mu]$ $(\mu>0)$ such that $F_\kappa(z',z_n)=x+z_n\nu_x$, $x=F_{\kappa,0}(z')$. Then F_κ is a diffeomorphism from B_μ onto $\bar{O}_\kappa=F_\kappa(B_\mu)$ for small μ, where O_κ is open in $\boldsymbol{R}^n$. Let Ψ be a pseudodifferential operator with $\sigma(\Psi)(\xi)=\psi(\xi)$, where ψ is defined in the previous example. Let $\theta_\kappa,\omega_\kappa\in C_0^\infty(O_\kappa)$ satisfy $\theta_\kappa\omega_\kappa=\omega_\kappa$ and $\omega_\kappa^* y_\lambda^\kappa=y_\lambda^\kappa$ on B, where $\omega_\kappa^*=F_\kappa^*\omega_\kappa$. Then we set

$$Z_\lambda^\kappa=F_\kappa^*\theta_\kappa K_\lambda F_\kappa^{-1*}\omega_\kappa.$$

Let P_λ be a pseudodifferential operator with $\sigma(P_\lambda)\in S^{m;k}(\boldsymbol{R}_z^n)$. Then we denote the single layer potential by

$$(P_{\lambda,z_n}w)(z')=(P_\lambda(\delta(y_n)\otimes w(y')))(z), \quad w\in\mathcal{S}'(\boldsymbol{R}^{n-1}).$$

Set $T_\lambda^\kappa=Z_{\lambda,0}^\kappa Y_\lambda^\kappa$. Then we can localize the operator T_λ.

Proposition 3. $T_\lambda-\sum_\kappa F_{\kappa,0}^{-1*}T_\lambda^\kappa F_{\kappa,0}^*\chi_\kappa\in S(-1)$.

Next we study $\sigma(T_\lambda^\kappa)$.

Proposition 4. *Set*

$$x_\lambda^\kappa(z',\zeta')=\frac{\omega_\kappa^*(z',0)}{2\pi}\int_{-\infty}^{\infty}(\psi k_\lambda)({}^t d_z F_\kappa^{-1}\zeta)|_{z_n=0}\,d\zeta_n,$$

where d_zF_κ denotes the Jacobi matrix of F_κ at z. Then we have

$$\sigma(Z_{\lambda,0}^\kappa)(z',\zeta')-x_\lambda^\kappa(z',\zeta')\in S^{0;-2}(\boldsymbol{R}^{n-1}).$$

Moreover, we have $\partial_{\zeta_j}x_\kappa^\lambda\in S^{0;-2}(\boldsymbol{R}^{n-1})$ for all j $(1\leqq j\leqq n-1)$.

By (Y1) and Proposition 2 we have

Proposition 5. $\sigma(T_\lambda^\kappa)=x_\lambda^\kappa y_\lambda^\kappa$ mod. $S^{0;-1}(\boldsymbol{R}^{n-1})$.

Let π_ν be the projection such that $\pi_\nu w=\langle\nu,w\rangle\nu$. Then, using Propositions 3 and 5, we can prove

Proposition 6. *We can choose y_λ^κ so that*

$$T_\lambda=(I-\pi_\nu)(I+J) \qquad \text{mod. } S(-1),$$

where J has a smooth kernel. In particular,

$$\langle\nu,T_\lambda\cdot\rangle\in S(-1).$$

We can take sufficient fine covering $\{U_\kappa\}$ so that

$$|Jw|_{L_r(S)}\leqq\frac{1}{2}|w|_{L_r(S)} \qquad \textit{for all } w\in \boldsymbol{L}_r(S).$$

Now we construct V_λ. We consider

$$W_\lambda g = K_\lambda(\delta_S \otimes Y_\lambda g) - \nabla N\langle \nu, T_\lambda g\rangle,$$

which satisfies (T) except the boundary condition $\gamma_S v = g$. Set $S_\lambda g = \gamma_S W_\lambda g$. Then it is clear that $\langle S_\lambda g, \nu\rangle = 0$. Take y_λ^ε as in Proposition 6. Then we have

Theorem 4. *The bounded linear operator*

$$S_\lambda : W^s_{r,\tau}(S) \longrightarrow W^s_{r,\tau}(S)$$

has the form

$$S_\lambda = I + (I - \pi_\nu)J \qquad \text{mod. } S(-1).$$

This implies S_λ has the inverse if $|\lambda|$ is large, so we have

Proposition 7. $V_\lambda g = W_\lambda S_\lambda^{-1} g$ *if* $|\lambda|$ *is large.*

By Propositions 1 and 7 we get

Theorem 5. $G_\lambda f = K_\lambda f - \nabla N\langle \nu, \gamma_S K_\lambda f\rangle - W_\lambda S_\lambda^{-1} M_\lambda f$ *if* $|\lambda|$ *is large.*

From this Theorem we can derive Theorems 1 and 2 (cf. Seeley [12]); we shall give the detailed proof elsewhere.

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