2. A Note on a Regularity of Irreducible Characters of a Non-Connected Lie Group

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(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1981)

1. Let G be a Lie group. We call a character of an irreducible unitary representation (IUR) of G an irreducible character of G. Little is studied about the regularity of irreducible characters when G is not connected. In this note, we present an interesting example from this point of view.

First recall some known facts. Let G be reductive, and assume that $\mathrm{Ad}(G)$ is contained in the connected complex adjoint group of $\mathfrak{g}_c$, where $\mathfrak{g}_c$ denotes the complexification of the Lie algebra $\mathfrak{g}$ of G. Then any irreducible character is a distribution on G which coincides with a locally summable function on G, analytic on an open dense subset of G (see [2, p. 132]). On the other hand, Shintani studied in [4] the relation between IURs of the group G generated by $G^0=SL(2, \boldsymbol{C})$ and σ, and those of $G_\sigma=SL(2, \boldsymbol{R})$, where σ is the complex conjugation of matrices. In this case, $G=G^0\cup G^0\sigma$, and $\mathrm{Ad}(\sigma)$ is an outer automorphism of $\mathfrak{g}_c$. Hence, to use the explicit forms of irreducible characters of G and those of G_σ, as an essential tool, he had to establish for G the similar result as above [4].

Now let G be nilpotent and connected. Then the character of an infinite-dimensional IUR is a distribution on G supported by a subvariety of lower dimension. In fact, by Kirillov's orbit method, such a representation is equivalent to that induced from a proper connected subgroup H by a unitary character of it. Therefore, the character of the representation is supported by the closure $\overline{G(H)}$ of the union $G(H)$ of gHg^{-1} over $g\in G$. Since H is proper, the normalizer N_G of H in G is strictly bigger than H. Put $\bar{g}(H)=gHg^{-1}$ for $\bar{g}=gN_G$, then $G(H)$ is the union of $\bar{g}(H)$ over $\bar{g}\in G/N_G$. Note that N_G is connected, and use the structure of a connected nilpotent Lie group given in [3, p. 83], then we see that $\dim \overline{G(H)}=\dim H+\dim (G/N_G)<\dim G$.

In these connections, we are interested in the following problems. Assume that G is unimodular, and that any irreducible character is a distribution on G, that is, for any IUR T of G, the operator

$$T(\varphi)=\int_G \varphi(g)T(g)dg \qquad \text{for } \varphi\in C_0^\infty(G)$$

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is of trace class, and the functional $\pi_T : \varphi \mapsto \mathrm{Tr}(T(\varphi))$, is a distribution on G. Here $C_0^\infty(G)$ denotes the space of all C^∞-functions on G with compact supports, and dg denotes a Haar measure on G. Then we ask what is a good sufficient condition for that, on a fixed connected component G^1 of G, every irreducible character of G coincides with a locally summable function on G^1. And further, is there any connection between this phenomenon on the connected component G^0 of 1 and that on $G^1 \neq G^0$?

Here we give only an example which suggests complexities of these problems.

2. Let G be the group given as the semidirect product of $\boldsymbol{R}^n$ and $O(n)$, the orthogonal group of order n, that is, $G=\boldsymbol{R}^n \times O(n)$, and $(x, u)(x', u')=(x+ux', uu')$ for $x, x' \in \boldsymbol{R}^n$, $u, u' \in O(n)$. The connected component G^0 of the neutral element of G is the group of Euclidean motions on $\boldsymbol{R}^n$. Put $K=O(n)$, $K^0=SO(n)$, and

$$s=\begin{pmatrix} 1_{n-1} & 0 \\ 0 & J \end{pmatrix} \quad \text{with} \quad J=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where 1_n denotes the unit matrix of degree n. Then $K=K^0 \cup K^0 s$, and $G=G^0 \cup G^0 s$. We prove the following

Theorem. *Let π be the character of an infinite-dimensional irreducible unitary representation of $G=\boldsymbol{R}^n \times O(n)$. Let n be odd. Then, on G^0, π coincides with a locally summable function which is analytic except on a subvariety of lower dimension, and on $G^0 s$, it is identically zero or supported by a subvariety of lower dimension. On the contrary, let n be even. Then, on G^0, π is supported by a subvariety of lower dimension, and on $G^0 s$, it coincides with a locally summable function which is analytic except on a subvariety of lower dimension.*

Explicit form of the characters π is also given in Lemmas 2 and 6.

3. We construct IURs of G by applying Mackey's theory of induced representations for an abelian extension of a group. Let $N=\boldsymbol{R}^n$ and $\hat{N}=\boldsymbol{R}^n$ be its dual. The duality is given by $\boldsymbol{R}^n \times \boldsymbol{R}^n \ni (x, \xi) \mapsto e^{i\langle x, \xi\rangle}$, where $\langle x, \xi\rangle = \sum_{1\leqslant j\leqslant n} x_j \xi_j$. The action of K on N induces naturally an action on $\hat{N}$ as $\xi \mapsto u\xi$ $(u \in K)$. Every orbit in $\hat{N}$ under K has a representative $\xi = {}^t(0, 0, \cdots, 0, r)$, $r \geqslant 0$. Infinite-dimensional IURs are constructed corresponding to non-trivial orbits as follows. Let $M=\{u \in K\,;\, u\xi=\xi\}$. Take an IUR ρ of M on a vector space V_ρ. Let $\mathscr{H}'$ be the space of V_ρ-valued continuous functions on K satisfying

$$(1) \qquad f(mk)=\rho(m)f(k) \qquad (m \in M,\ k \in K).$$

We put

$$(2) \qquad \|f\|^2=\int_K \|f(k)\|_{V_\rho}^2\, dk,$$

where dk denotes the Haar measure on K normalized as $\int_K dk=1$.

Denote by $\mathcal{H}$ the completion of $\mathcal{H}'$ with respect to $\|f\|$. Then the representation T induced from $N\times M$ by $L:(x, m)\mapsto e^{i\langle x,\xi\rangle}\rho(m)$, is realized on $\mathcal{H}$ by right translations:

(3) $$T(x, u)f(k)=e^{i\langle kx,\xi\rangle}f(ku) \qquad ((x, u)\in G,\ f\in\mathcal{H}).$$

Every infinite-dimensional IUR of G is equivalent to T for some ξ and ρ.

4. For $\varphi\in C_0^\infty(G)$, the operator $T(\varphi)$ is given by

$$T(\varphi)f(k)=\int_{G=N\times K} e^{i\langle kx,\xi\rangle}f(ku)\,\varphi(x, u)\,dx\,du,$$

where dx denotes the usual Lebesgue measure on $N=\boldsymbol{R}^n$. Then $T(\varphi)$ is an integral operator on $\mathcal{H}$ with a kernel K_φ:

$$T(\varphi)f(k)=\int_K K_\varphi(k, k')f(k')\,dk',$$

where

$$K_\varphi(k, k')=\int_N e^{i\langle kx,\xi\rangle}\varphi(x, k^{-1}k')dx.$$

To calculate the trace of $T(\varphi)$, we proceed as follows. Let $L^2(K, V_\rho)$ be the space of V_ρ-valued measurable functions f on K such that $\|f\|<+\infty$, and let P be the orthogonal projection of $L^2(K, V_\rho)$ onto $\mathcal{H}$. Then for $f\in L^2(K, V_\rho)$,

$$(Pf)(k)=\int_M \rho(m)f(m^{-1}k)dm,$$

where dm denotes the normalized Haar measure on M. Let S be the integral operator on $L^2(K, V_\rho)$ with the kernel K_φ. Then $\mathrm{Tr}\,(T(\varphi))=\mathrm{Tr}\,(PS)$, and the kernel K'_φ of PS is given by

$$K'_\varphi(k, k')=\int_M \rho(m)K_\varphi(m^{-1}k, k')dm.$$

Since the kernel K'_φ is of class C^∞ on $K\times K$, we have

$$\mathrm{Tr}(PS)=\int_K \mathrm{Tr}(K'(k, k))\,dk=\int_K\int_M \pi_\rho(m)K_\varphi(m^{-1}k, k)dm\,dk,$$

where $\pi_\rho(m)=\mathrm{Tr}(\rho(m))$ is the character of ρ. Therefore,

$$\mathrm{Tr}(T(\varphi))=\int_N\int_K\int_M \pi_\rho(m)e^{i\langle kx,\xi\rangle}\varphi(x, k^{-1}mk)dm\,dk\,dx.$$

This gives the character π_T of T as a functional in φ, which is expressed formally as

(4) $$\mathrm{Tr}(T(\varphi))=\int_N\int_K \varphi(x, u)\pi_T(x, u)du\,dx.$$

We wish to study the support of the distribution π_T, and to calculate $\pi_T(x, u)$ on a subdomain of G where it coincides with a locally summable function.

Put $K^1=K^0s$, $G^1=G^0s$, $M^0=M\cap K^0$, and $M^1=M\cap K^1=M^0s$. Denote by d^0m and d^0k the normalized Haar measures on M^0 and K^0 respectively. Then, for $\varphi\in C_0^\infty(G)$ with support contained in $\mathrm{G}^p=N\times K^p$ for $p=0$ or 1, we have respectively

$$\mathrm{Tr}(T(\varphi))=\frac{1}{2}\int_N\int_K\int_{M^0}\pi_\rho(ms^p)e^{i\langle kx,\xi\rangle}\varphi(x,k^{-1}ms^pk)d^0m\,dk\,dx.$$

Put $t=\begin{pmatrix}1_{n-1} & 0\\ 0 & -1\end{pmatrix}\in K^1$, then $-t\in M$. Note that $\pi_\rho(t^{-1}mt)=\pi_\rho(m)$, and $\langle tkx,\xi\rangle=-\langle kx,\xi\rangle$. Then the above formula is rewritten as

$$(5)\qquad \mathrm{Tr}(T(\varphi))=\frac{1}{4}\int_N\int_{K^0}\int_{M^0}\pi_\rho(ms^p)(e^{i\langle kx,\xi\rangle}+e^{-i\langle kx,\xi\rangle})\times\varphi(x,k^{-1}ms^pk)d^0m\,d^0k\,dx.$$

Clearly, on G^p, for $p=0$ or 1, the support of π_T is contained in $N\times\bigcup_{k\in K^0}k^{-1}M^pk$.

5. Case of $n=2q+1$ (odd). First consider π_T on G^0. Let H^0 be a subgroup of $M^0\cong SO(n-1)$ consisting of elements of the form

$$(6)\qquad h=\begin{bmatrix}u(\theta_1) & & & & \\ & u(\theta_2) & & 0 & \\ & & \ddots & & \\ & 0 & & u(\theta_q) & \\ & & & & 1\end{bmatrix}\quad\text{with } u(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

Then H^0 is a Cartan subgroup of M^0 and that of K^0 at the same time. Put $\lambda_j=e^{i\theta_j}(1\leqslant j\leqslant q)$, and

$$D_M(h)=\prod_{j\neq k}(1-\lambda_j\lambda_k^{-1})\prod_{j<k}(1-\lambda_j\lambda_k)(1-\lambda_j^{-1}\lambda_k^{-1}),$$
$$D_K(h)=D_M(h)\prod_{1\leqslant j\leqslant q}(1-\lambda_j)(1-\lambda_j^{-1}).$$

Then the Weyl's integration formulas for M^0 and K^0 are given as follows.

Lemma 1. *Let ψ and ψ' be continuous functions on M^0 and K^0 respectively. Then*

$$\int_{M^0}\psi(m)\,d^0m=\frac{1}{|W_M|}\int_{H^0}\int_{M^0}\psi(m^{-1}hm)\,D_M(h)\,d^0m\,dh,$$
$$\int_{K^0}\psi'(k)d^0k=\frac{1}{|W_K|}\int_{H^0}\int_{K^0}\psi'(k^{-1}hk)\,D_K(h)\,d^0k\,dh,$$

where dh denotes the normalized Haar measure on H^0, and W_M, W_K denote the Weyl groups of (M^0,H^0), (K^0,H^0) respectively.

Apply this lemma to (5) with $p=0$, and note that W_K/W_M has a system of representatives $\{1,\omega\}$, where ω changes only the sign of θ_1. Then we get the following

Lemma 2. *For $(x,u)\in G^0$, let $u=k^{-1}hk$ with $h\in H^0$, $k\in K^0$, and put*

$$\pi_T(x,u)=\frac{1}{2}(\pi_\rho(h)+\pi_\rho(\omega h))\frac{D_M(h)}{D_K(h)}(e^{i\langle kx,\xi\rangle}+e^{-i\langle kx,\xi\rangle}).$$

Then this function is locally summable and gives π_T on G^0.

Next consider π_T on G^1. Let H' be a subgroup of H^0 consisting of elements h in (6) with $\theta_q=0$.

Lemma 3. *Any element of* $M^1=M^0s$ *is conjugate to some* $h's \in H's$. *Any element in* $K^1=K^0s=K^0t$ *is conjugate to some* $ht\in H^0t$. *Two elements* h_1t *and* h_2t *are conjugate to each other under* K^0 *if and only if so are* h_1 *and* h_2 *under* W_K. *Moreover* $h's\in H's$ *is conjugate to* $h't\in H^0t$ *under* K^0 *for any* $h'\in H'$.

From this lemma, we see that π_T on G^1 is a distribution supported by a subvariety of lower dimension.

6. Case of $n=2q$ (even). First consider π_T on G^0. Let H^0 (resp. A^0) be a subgroup of $M^0\cong SO(n-1)$ (resp. of K^0) consisting of elements of the form

$$h=\begin{bmatrix} u(\theta_1) & & & & 0 \\ & u(\theta_2) & & & \\ & & \ddots & & \\ & & & u(\theta_{q-1}) & \\ 0 & & & & 1_2 \end{bmatrix} \left(\text{resp. } a=\begin{pmatrix}1_{n-2} & 0 \\ 0 & u(\theta_q)\end{pmatrix}\right). \tag{7}$$

Then H^0 is a Cartan subgroup of M^0, and H^0A^0 is that of K^0. This shows that π_T on G^0 is a distribution supported by a subvariety of lower dimension.

Next consider π_T on G^1.

Lemma 4. *Any element in* $M^1=M^0s$ *is conjugate to some* $hs\in H^0s$. *Similarly any element in* $K^1=K^0s$ *is conjugate to some* $hs\in H^0s$.

Let Z^- be the centralizer of H^0s in K^0, and $N^-=\{k\in K^0;\ k(H^0s)k^{-1}\subset H^0s\}$. Then $Z^-=H^0\cup H^0s_0$, where s_0 is a in (7) with $\theta_q=\pi$. The group $W_-=N^-/Z^-$ is isomorphic to the Weyl group of (M^0, H^0), and moreover acts on H^0s in the similar way. Put for h in (7), $\lambda_j=e^{i\theta_j}$ $(1\leqslant j\leqslant q-1)$ and

$$D_M^-(hs)=\prod_{j\neq k}(1-\lambda_j\lambda_k^{-1})\prod_{j<k}(1-\lambda_j\lambda_k)(1-\lambda_j^{-1}\lambda_k^{-1})\prod_{1\leqslant j\leqslant q-1}(1+\lambda_j)(1+\lambda_j^{-1}),$$

$$D_K^-(hs)=D_M^-(h)\prod_{1\leqslant j\leqslant q-1}(1-\lambda_j)(1-\lambda_j^{-1}).$$

Then we have the following integration formulas.

Lemma 5. *Let* ψ *and* ψ' *be continuous functions on* M^1 *and* K^1 *respectively. Then*

$$\int_{M^0}\psi(ms)d^0m=\frac{1}{|W_-|}\int_{H^0}\int_{M^0}\psi(m^{-1}hsm)D_M^-(hs)d^0m\,dh,$$

$$\int_{K^0}\psi'(ks)d^0k=\frac{1}{|W_-|}\int_{H^0}\int_{K^0}\psi'(k^{-1}hsk)D_K^-(hs)d^0k\,dh.$$

We note here that a Cartan subgroup of non-connected K in the sense of Chevalley [1, Exposé 7] turns out, in this case, to be the centralizer of a maximal torus in K, and hence is contained completely in K^0. For instance, H^0A^0 is a Cartan subgroup of both K^0 and K at the same time.

Apply the above integration formulas to (5) with $p=1$ successively, we get the following

Lemma 6. *For* $(x, us) \in G^1$, *let* $us = k^{-1}hsk$ *with* $h \in H^0$, $k \in K^0$, *and put*

$$\pi_T(x, us) = \frac{1}{2}\pi_\rho(hs)\frac{D_M^-(hs)}{D_K^-(hs)}(e^{i\langle kx,\xi\rangle} + e^{-i\langle kx,\xi\rangle}).$$

Then this function is locally summable and gives π_T on G^1.

Remark. The regularity of irreducible characters on the connected component containing the space reflexion of $E(2) = R^2 \times O(2)$ (the case $n=2$), and especially of the (non-connected) Heisenberg group is utilized in Wigner's formula for the inverse of the Weyl quantization map [5].

The second named author dedicates this paper to his friend the late Shintani, eventhough it is too small for his originality in mathematics which he admired all the time. The conversation with him about his work [4] was one of the motivations to study the subject in this paper.

References

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