15. On Certain Numerical Invariants of Mappings over Finite Fields. IV

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Introduction. This is a continuation of our papers [2], [3] which will be referred to as (I), (II), respectively, in this paper.[1] Let K be a field of characteristic not 2 and X be a composition algebra over K. By definition, X is an algebra (not necessarily associative) with 1 over K together with a nonsingular quadratic form q_X such that $q_X(xy) = q_X(x)q_X(y)$, $x, y \in X$. Thanks to a theorem due to Hurwitz (cf. [1], Theorem 3.25, p. 73), such algebras are completely determined. Namely, an algebra (X, q_X) is one of the following: (I) $X=K$; (II) $X=K\oplus K$; (III) $X=a$ quadratic extension of K; (IV) $X=a$ quaternion algebra over K; (V) $X=a$ Cayley algebra over K. Furthermore, if $X=K$, then $q_X(x)=x^2$; otherwise q_X is the norm form on X. Therefore, we shall put $n(x)=q_X(x)$.

From now on, assume that $K=F_q$, the finite field with q (odd) elements. Then the composition algebras $(X, n(x)=\bar{x}x)$ can be described more precisely as follows:

(I) $X=K$, $\bar{x}=x$, $n(x)=x^2$,

(II) $X=K\oplus K$, $\bar{x}=(x_2, x_1)$ if $x=(x_1, x_2)$, and $n(x)=x_1x_2$,

(III) $X=F_{q^2}=$the unique quadratic extension of K, $\bar{x}=$the conjugate of x, $n(x)=\bar{x}\,x$,

(IV) $X=K_2=$the algeba of matrices of order 2, $\bar{x}=\begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$ if $x=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, and $n(x)=\bar{x}\,x=\det x$,

(V) $X=K_2\times K_2$ with the multiplication $zw=(xu+\bar{v}y, vx+y\bar{u})$ for $z=(x, y)$, $w=(u, v)\in X$ where $\bar{x}=$the adjoint of $x\in K_2$ defined above, and $\bar{z}=(\bar{x}, -y)$, $n(z)=n(x)-n(y)=\det x-\det y$.

The Hopf map F associated to a composition algebra (X, n) is the map

(0.1) $F: Z=X\times X\to W=K\times X$ defined by
$F(z)=(n(x)-n(y), 2xy)$.

Our purpose is to determine the invariants σ_F, ρ_F introduced in the paper (I) for the map F with respect to the quadratic character χ of K.

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1) For example, we mean by (II.3.4) the item (3.4) in (II).

§ 1. Statement of the results. Let $K=F_q$ (q : odd) and let F be a quadratic mapping $X \to Y$ of vector spaces over K, $n=\dim X$, $m=\dim Y$. For each $\lambda \in Y^*$ (the dual of Y) put $F_\lambda=\lambda \circ F$, a quadratic form on X. Denote by r_λ the rank of F_λ. Put

(1.1) $S_{F_\lambda}=\sum_{x \in X} \chi(F_\lambda(x))$,

where χ means the quadratic character of the multiplicative group $K^\times$ (extended by $\chi(0)=0$). It is known that (cf. (II.1.3), (II.1.4))

(1.2) $S_{F_\lambda}=\begin{cases} 0, & \text{if } r_\lambda \text{ is even}, \\ q^{n-((r_\lambda+1)/2)}(q-1)\chi((-1)^{(r_\lambda-1)/2}d_\lambda), & \text{if } r_\lambda \text{ is odd}, \end{cases}$

where $d_\lambda=\det F_\lambda$. From (1.2) it follows that

(1.3) $\sigma_F \overset{\text{def}}{=} \sum_{\lambda \in Y^*} |S_{F_\lambda}|^2=(q-1)^2 \sum_{r_\lambda \text{ odd}} q^{2n-r_\lambda-1}$.

By (I.1.11), (1.3), we have

(1.4) $\rho_F \overset{\text{def}}{=} \sum_{(x,y) \in P} \chi(F(x):F(y))$[2] $=q^{n-m}(q-1) \sum_{r_\lambda \text{ odd}} q^{n-r_\lambda}$.

(1.5) **Theorem.** *Let X be a composition algebra over the field $K=F_q$ (q : odd), and $F: X \times X \to K \times X$ be the associated Hopf map. Then, we have $S_{F_\lambda}=0$ for all $\lambda \in (K \times X)^*$ (and hence $\sigma_F=\rho_F=0$) except for the case* (I) $X=K$, $q \equiv 1 \pmod 4$, *and in the latter case $\sigma_F=2q^2(q-1)^3$, $\rho_F=2q(q-1)^2$.*

Our proof of the theorem splits into five parts according to the classification.

§ 2. Type (I). In this case, we have $X=K$, $n(x)=x^2$, $Z=X^2=K^2$, $W=K \times X=K^2$ and $F(z)=(x^2-y^2, 2xy)$. Hence, $F_\lambda(z)=\lambda_1(x^2-y^2)+\lambda_2(2xy)$ $=\lambda_1 x^2+2\lambda_2 xy-\lambda_1 y^2$ and the corresponding matrix is

(2.1) $\Phi_\lambda=\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & -\lambda_1 \end{pmatrix}$.

When $\lambda \neq 0$, we have $r_\lambda=\operatorname{rank} \Phi_\lambda=1$ or 2 according as $\lambda_2^1+\lambda_2^2=0$ or $\neq 0$. Therefore $r_\lambda=2$ always when $q \equiv 3 \pmod 4$. On the other hand, when $q \equiv 1 \pmod 4$, since the number of $\lambda=(\lambda_1, \lambda_2) \neq 0$ with $\lambda_1^2+\lambda_2^2=0$ is $2(q-1)$, there are $2(q-1)$ λ's for which $r_\lambda=1$. For each such λ we have, by (1.2), $|S_{F_\lambda}|=q(q-1)$ and so $\sigma_F=2q^2(q-1)^3$, $\rho_F=2q(q-1)^2$ by (1.3), (1.4).

§ 3. Type (II). In this case, we have $X=K \oplus K$, $n(x)=x_1x_2$ for $x=(x_1, x_2) \in X$, $Z=X \times X$, $W=K \times X$ and $F(z)=(n(x)-n(y), 2xy)$ where $xy=(x_1y_1, x_2y_2)$. Hence, if we put $\lambda_1=\gamma \in K$, $\lambda'=(\alpha, \beta) \in X^*$, we have $F_\lambda(z)=\lambda_1(n(x)-n(y))+2\lambda'(xy)=\gamma(x_1x_2-y_1y_2)+2\alpha x_1y_1+2\beta x_2y_2$ and the corresponding matrix is

(3.1) $\Phi_\lambda=\begin{pmatrix} \frac{\gamma}{2}J & A \\ A & -\frac{\gamma}{2}J \end{pmatrix}$

2) As for the meaning of unexplained notations, see (I. § 1).

with

(3.2) $J=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad A=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}.$

Observe that $r_\lambda=4-\dim(\operatorname{Ker}\Phi_\lambda)$. Therefore, if $\gamma=0$, then $\dim(\operatorname{Ker}\Phi_\lambda)=2\dim(\operatorname{Ker}A)\equiv 0 \pmod 2$, hence $r_\lambda\equiv 0 \pmod 2$. On the other hand, if $\gamma\neq 0$, since we have

(3.3) $\Phi_\lambda\begin{pmatrix}x\\ y\end{pmatrix}=0 \iff \begin{cases}\dfrac{\gamma}{2}Jx+Ay=0,\\ \dfrac{\gamma}{2}Jy-Ax=0,\end{cases}$

on eliminating y, we have

(3.4) $(\gamma^2 J+4AJA)x=0.$

Since a simple computation shows that

(3.5) $AJA=\alpha\beta J,$

(3.4) is equivalent to the trivial equation

(3.6) $(\gamma^2+4\alpha\beta)u=0,$

which implies that $r_\lambda=\operatorname{rank}\Phi_\lambda=4-\dim(\operatorname{Ker}\Phi_\lambda)=4-(0\text{ or }2)\equiv 0 \pmod 2$, again.

§4. Type (III). In this case, we have $X=F_{q^2}=K(\theta)$, $\theta^2=m\in K$, $n(x)=x_1^2-mx_2^2$ if $x=x_1+\theta x_2$, $Z=X\times X$, $W=K\times X$ and $F(z)=(n(x)-n(y), 2xy)$ where $xy=(x_1y_1+mx_2y_2)+(x_1y_2+x_2y_1)\theta$. Hence, if we put $\lambda_1=\gamma\in K$, $\lambda'=(\alpha,\beta)\in X^*$, we have $F_\lambda(z)=\lambda_1(n(x)-n(y))+2\lambda'(xy)=\gamma(x_1^2-mx_2^2-y_1^2+my_2^2)+2\alpha(x_1y_1+mx_2y_2)+2\beta(x_1y_2+x_2y_1)$ and the corresponding matrix is

(4.1) $\Phi_\lambda=\begin{pmatrix}\gamma J & A\\ A & -\gamma J\end{pmatrix}$

with

(4.2) $J=\begin{pmatrix}1 & 0\\ 0 & -m\end{pmatrix}, \quad A=\begin{pmatrix}\alpha & \beta\\ \beta & m\alpha\end{pmatrix}.$

Observe that $r_\lambda=4-\dim(\operatorname{Ker}\Phi_\lambda)$. Therefore, if $\gamma=0$, then $\dim(\operatorname{Ker}\Phi_\lambda)=2\dim(\operatorname{Ker}A)$ and so $r_\lambda\equiv 0 \pmod 2$. On the other hand, if $\gamma\neq 0$, since we have

(4.3) $\Phi_\lambda\begin{pmatrix}x\\ y\end{pmatrix}=0 \iff \begin{cases}\gamma Jx+Ay=0,\\ \gamma Jy-Ax=0,\end{cases}$

we have

(4.4) $(\gamma^2 J+AJ^{-1}A)x=0.$

Since a simple computation shows that

(4.5) $AJ^{-1}A=\left(\alpha^2-\dfrac{\beta^2}{m}\right)J,$

(4.4) is equivalent to the trivial equation

(4.6) $\left(\gamma^2+\left(\alpha^2-\dfrac{\beta^2}{m}\right)\right)u=0,$

which implies that $r_\lambda = \text{rank}\,\Phi_\lambda = 4 - \dim(\text{Ker}\,\Phi_\lambda) = 4 - (0 \text{ or } 2) \equiv 0 \pmod 2$, again.

§ 5. Type (IV). In this case, we have $X = K_2$, $n(x) = \det x$, $Z = X \times X$, $W = K \times X$ and $F(z) = (n(x) - n(y), 2xy)$ where xy is the matrix multiplication. Hence, if we put $\lambda_1 = \gamma \in K$, $\lambda' = \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in X^*$ with $\alpha x = \sum_{i=1}^{4} \alpha_i x_i$, we have $F_\lambda(z) = \lambda_1(n(x) - n(y)) + 2\lambda'(xy) = \gamma(x_1x_4 - x_2x_3 - y_1y_4 + y_2y_3) + 2\alpha_1(x_1y_1 + x_2y_3) + 2\alpha_2(x_1y_2 + x_2y_4) + 2\alpha_3(x_3y_1 + x_4y_3) + 2\alpha_4(x_3y_2 + x_4y_4)$ and the corresponding matrix is

$$(5.1) \quad \Phi_\lambda = \begin{pmatrix} \frac{\gamma}{2} J & A \\ {}^tA & -\frac{\gamma}{2} J \end{pmatrix}$$

with

$$(5.2) \quad J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & 0 & \alpha_3 & \alpha_4 \end{pmatrix}.$$

Observe that $r_\lambda = 8 - \dim(\text{Ker}\,\Phi_\lambda)$. Therefore, if $\gamma = 0$, then $\dim(\text{Ker}\,\Phi_\lambda) = \dim(\text{Ker}\,A) + \dim(\text{Ker}\,{}^tA) = 2\dim(\text{Ker}\,A)$ and so $r_\lambda \equiv 0 \pmod 2$. On the other hand, if $\gamma \neq 0$, since we have

$$(5.3) \quad \Phi_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff \begin{cases} \frac{\gamma}{2} Jx + Ay = 0, \\ \frac{\gamma}{2} Jy - {}^tAx = 0, \end{cases}$$

we have

$$(5.4) \quad (\gamma^2 J + 4AJ\,{}^tA)x = 0.$$

Since a simple computation shows that

$$(5.5) \quad AJ\,{}^tA = (\det \alpha)J,$$

(5.4) is equivalent to the trivial equation

$$(5.6) \quad (\gamma^2 + 4\det\alpha)u = 0,$$

which implies that $r_\lambda = \text{rank}\,\Phi_\lambda = 8 - \dim(\text{Ker}\,\Phi_\lambda) = 8 - (0 \text{ or } 4) \equiv 0 \pmod 2$, again.

§ 6. Type (V). In this case, we have $X = K_2 \times K_2$, $n(z) = \det x - \det y$ if $z = (x, y) = X$, $Z = X \times X$, $W = K \times X$ and $F(z, w) = (n(z) - n(w), 2zw)$ where $z = (x, y)$, $w = (u, v) \in Z = X \times X$ and $zw = (xu + \bar{v}y, vx + y\bar{u})$. Hence, if we put $\lambda_1 = \gamma \in K$, $\lambda' = (\alpha, \beta) \in X^*$ with $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$, $\beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$, we have $F_\lambda(z, w) = \lambda_1(n(z) - n(w)) + 2\lambda'(zw) = \gamma(\det x - \det y - \det u + \det v) + 2\alpha(xu + \bar{v}y) + 2\beta(vx + y\bar{u})$[3] and the corresponding matrix is

3) We here omit the expression in terms of coordinates because it is too long.

$$(6.1)\quad \Phi_\lambda=\begin{pmatrix} \frac{\gamma}{2}J & A \\ {}^tA & -\frac{\gamma}{2}J \end{pmatrix}$$

with

$$(6.2)\quad J=\left[\begin{array}{cccc|cccc} & & & 1 & & & & \\ & & -1 & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ \hline & & & & & & & -1 \\ & & & & & & 1 & \\ & & & & & 1 & & \\ & & & & -1 & & & \end{array}\right],$$

$$A=\left[\begin{array}{cccc|cccc} \alpha_1 & \alpha_2 & 0 & 0 & \beta_1 & 0 & \beta_3 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & \beta_2 & 0 & \beta_4 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 & 0 & \beta_1 & 0 & \beta_3 \\ 0 & 0 & \alpha_3 & \alpha_4 & 0 & \beta_2 & 0 & \beta_4 \\ \hline 0 & -\beta_2 & 0 & \beta_1 & 0 & 0 & -\alpha_3 & \alpha_1 \\ \beta_2 & 0 & -\beta_1 & 0 & 0 & 0 & -\alpha_4 & \alpha_2 \\ 0 & -\beta_4 & 0 & \beta_3 & \alpha_3 & -\alpha_1 & 0 & 0 \\ \beta_4 & 0 & -\beta_3 & 0 & \alpha_4 & -\alpha_2 & 0 & 0 \end{array}\right].$$

Observe that $r_\lambda=16-\dim(\mathrm{Ker}\,\Phi_\lambda)$. Therefore, if $\gamma=0$, then $\dim(\mathrm{Ker}\,\Phi_\lambda)=\dim(\mathrm{Ker}\,A)+\dim(\mathrm{Ker}\,{}^tA)=2\dim(\mathrm{Ker}\,A)$ and so $r_\lambda\equiv 0 \pmod 2$. On the other hand, if $\gamma\neq 0$, since we have

$$(6.3)\quad \Phi_\lambda\begin{pmatrix} x \\ y \end{pmatrix}=0 \iff \begin{cases} \frac{\gamma}{2}Jx+Ay=0, \\ \frac{\gamma}{2}Jy-{}^tAx=0, \end{cases}$$

we have

$$(6.4)\quad (\gamma^2J+4AJ{}^tA)x=0.$$

Since a simple (but lengthy) computation shows that

$$(6.5)\quad AJ{}^tA=(\det\alpha-\det\beta)J,$$

(6.4) is equivalent to the trivial equation

$$(6.6)\quad (\gamma^2+4(\det\alpha-\det\beta))u=0,$$

which implies that $r_\lambda=\mathrm{rank}\,\Phi_\lambda=16-\dim(\mathrm{Ker}\,\Phi_\lambda)=16-(0\text{ or }8)\equiv 0 \pmod 2$, again. Q.E.D.

References

[1] Schafer, R.: An Introduction to Nonassociative Algebras. Academic Press, New York (1966).
[2] Ono, T.: On certain numerical invariants of mappings over finite fields. I. Proc. Japan Acad., **56A**, 342–347 (1980).
[3] ——: ditto. II. ibid., **56A**, 397–400 (1980).