11. *On Eisenstein Series for Siegel Modular Groups*

By Nobushige KUROKAWA
Department of Mathematics, Tokyo Institute of Technology

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Introduction. We present some results on Eisenstein series for Siegel modular groups. These results concern the action of Hecke operators and the Fourier coefficients. We refer to [3] for the motivation of these results. We use the notations of [4].

§ 1. Eisenstein series. For integers $n\geqq 0$ and $k\geqq 0$, we denote by $M_k(\Gamma_n)$ (resp. $S_k(\Gamma_n)$) the $\boldsymbol{C}$-vector space of all Siegel modular (resp. cusp) forms of degree n and weight k. (See [4, § 3] for Siegel modular forms of degree zero.) The space of Eisenstein series is $E_k(\Gamma_n)=S_k(\Gamma_n)^{\perp}$, which is the orthogonal complement of $S_k(\Gamma_n)$ in $M_k(\Gamma_n)$ with respect to the Petersson inner product $\langle\ ,\ \rangle$. For each even integer $k>2n$, the space $E_k(\Gamma_n)$ is constructed from $M_k(\Gamma_{n-1})$ by using the Eisenstein series of Langlands [5] and Klingen [1]. To be precise we define a $\boldsymbol{C}$-linear map $[\quad]^{(n-r)}: M_k(\Gamma_r)\to M_k(\Gamma_n)$ for $0\leqq r\leqq n$ and even $k>n+r+1$ as follows. Each modular form f in $M_k(\Gamma_r)$ is written uniquely as $f=\sum_{j=0}^r E_{r,j}^k(*,f_j)$ with cusp forms $f_j\in S_k(\Gamma_j)$ $(0\leqq j\leqq r)$, where $E_{r,j}^k(*,f_j)$ is the Eisenstein series defined in Klingen [1]. We define $[f]^{(n-r)}=\sum_{j=0}^r E_{n,j}^k(*,f_j)$. Then $[f]^{(n-r)}$ is a modular form in $M_k(\Gamma_n)$ satisfying $\Phi^{n-r}([f]^{(n-r)})=f$, where Φ is the Siegel operator. In particular, $[\quad]^{(0)}$ is the identity map, and we write $[\quad]=[\quad]^{(1)}$ for simplicity. Then it holds that

$$E_k(\Gamma_n)=[M_k(\Gamma_{n-1})]=\text{Image}\,([\quad]: M_k(\Gamma_{n-1})\to M_k(\Gamma_n))$$

for $n\geqq 1$ and even $k>2n$. More precisely we have $E_k^r(\Gamma_n)=[S_k(\Gamma_r)]^{(n-r)}$ and $\oplus_{j=0}^r E_k^j(\Gamma_n)=[M_k(\Gamma_r)]^{(n-r)}$ for $0\leqq r\leqq n$ and even $k>n+r+1$, where $E_k^j(\Gamma_n)=\mathfrak{S}_{kj}^{(n)}$ in the notation of Maass [6]. For $0\leqq r\leqq n$ and even $k>2n$, $[\quad]^{(n-r)}$ is the following $(n-r)$-times composition of $[\quad]$ ("$(n-r)$-th power"):

$$M_k(\Gamma_r)\xrightarrow{[\ ]}M_k(\Gamma_{r+1})\xrightarrow{[\ ]}\cdots\xrightarrow{[\ ]}M_k(\Gamma_n).$$

We use also the following extended definition: if $f\in M_k(\Gamma_r)$, $r\leqq j\leqq n$, $k>n+r+1$ even, and $F=[f]^{(j-r)}$, then we define that $[F]^{(n-j)}=[f]^{(n-r)}$.

Theorem 1. *Let f be an eigen modular form in $M_k(\Gamma_r)$ for $r\geqq 0$ and even $k>n+r+1$ with $n\geqq r$. Then $[f]^{(n-r)}$ is an eigen modular form in $M_k(\Gamma_n)$.*

Proof. In this proof j runs over $j=0,\cdots,r$. Write $f=\sum_j [f_j]^{(r-j)}$ with $f_j\in S_k(\Gamma_j)$, then $[f]^{(n-r)}=\sum_j [f_j]^{(n-j)}\in\oplus_{j=0}^r E_k^j(\Gamma_n)$. Take a Hecke operator $T\in \boldsymbol{T}(M_k(\Gamma_n))$. Since $E_k^j(\Gamma_n)$ is stable under $\boldsymbol{T}(M_k(\Gamma_n))$, we have $T[f]^{(n-r)}=\sum_j [g_j]^{(n-j)}$ with $g_j\in S_k(\Gamma_j)$, hence

$$\Phi^{n-r}T[f]^{(n-r)}=\sum_j [g_j]^{(r-j)}.$$

Denoting by T^* the image of T under the surjective map $\boldsymbol{T}(M_k(\Gamma_n))\to\boldsymbol{T}(M_k(\Gamma_r))$ constructed by Maass [6] and Zharkovskaya [13], we have $\Phi^{n-r}T[f]^{(n-r)}=T^*\Phi^{n-r}[f]^{(n-r)}=T^*f=\lambda(T^*,f)f$. Hence we have

$$\sum_j [g_j-\lambda(T^*,f)f_j]^{(r-j)}=0,$$

so $g_j=\lambda(T^*,f)f_j$ and $T[f]^{(n-r)}=\lambda(T^*,f)[f]^{(n-r)}$. Hence $[f]^{(n-r)}$ is an eigen modular form in $M_k(\Gamma_n)$ with the eigenvalue $\lambda(T,[f]^{(n-r)})=\lambda(T^*,f)$ for each $T\in\boldsymbol{T}(M_k(\Gamma_n))$. Q.E.D.

Remark 1. Let F be an eigen modular form in $M_k(\Gamma_n)$ for $n\geqq 0$ and $k\geqq 0$, and assume that $\Phi^{n-r}F=f\neq 0$ for an integer r in $0\leqq r\leqq n$. Then f is an eigen modular form in $M_k(\Gamma_r)$ and it holds that $\lambda(T,F)=\lambda(T^*,f)$ for all $T\in\boldsymbol{T}(M_k(\Gamma_n))$. In fact, from $TF=\lambda(T,F)F$ we have $T^*f=T^*\Phi^{n-r}F=\Phi^{n-r}TF=\lambda(T,F)\Phi^{n-r}F=\lambda(T,F)f$. In particular we have $\lambda(p,F)=\lambda(p,f)\prod_{j=r+1}^{n}(1+p^{k-j})$ for all prime numbers p; see Maass [6] and Zharkovskaya [13].

§2. A characterization of Eisenstein series. We prove a uniqueness property of Eisenstein series in a special case.

Theorem 2. *Let f be an eigen modular form in $M_k(\Gamma_1)$. Let $n\geqq 1$ be an integer, and assume that k is an even integer larger than* $n+1$ (resp. $n+2$) *if $\Phi f\neq 0$* (resp. $\Phi f=0$). *Then*:

(1) $m(\lambda([f]^{(n-1)}))=1$.

(2) *Let F be an eigen modular form in $M_k(\Gamma_n)$ satisfying $\lambda(F)=\lambda([f]^{(n-1)})$, then there exists a non-zero constant $\gamma\in\boldsymbol{C}$ such that $F=\gamma[f]^{(n-1)}$.*

(3) *Let F be an eigen modular form in $M_k(\Gamma_n)$ satisfying $\Phi^{n-1}(F)=f$, then $F=[f]^{(n-1)}$.*

Proof. (1) is equivalent to (2), and (3) follows from (2) by Remark 1 in §1 (from $\Phi^{n-1}(F)=f$ we have $\lambda(F)=\lambda([f]^{(n-1)})$ by Remark 1, hence $F=\gamma[f]^{(n-1)}$ by (2), then $\Phi^{n-1}(F)=\gamma f$, so $\gamma=1$). Hence it is sufficient to prove (2). Since the case $n=1$ is well-known (the multiplicity one theorem for $M_k(\Gamma_1)$), we assume that $n\geqq 2$ hereafter.

We prove the following refinement of (2):

(2*) *Let F be an eigen modular form in $M_k(\Gamma_n)$ satisfying $\lambda(p,F)=\lambda(p,[f]^{(n-1)})$ for all prime numbers p, then there exists a non-zero constant $\gamma\in\boldsymbol{C}$ such that $F=\gamma[f]^{(n-1)}$.*

Hereafter in this proof, p runs over all prime numbers.

We show first that $\Phi^{n-1}(F)\neq 0$. Suppose that $\Phi^{n-1}(F)=0$. Let r be the maximal integer $\leqq n$ such that $\Phi^{n-r}(F)\neq 0$. Then $h=\Phi^{n-r}(F)$ is an eigen cusp form in $S_k(\Gamma_r)$ and $2\leqq r\leqq n$. By Theorem 1 and Remark 1, we have

$$\lambda(p, F)=\lambda(p, h) \prod_{j=r+1}^{n} (1+p^{k-j})$$

and

$$\lambda(p, [f]^{(n-1)})=\lambda(p, f) \prod_{j=2}^{n} (1+p^{k-j}).$$

Hence, from the assumption $\lambda(p, F)=\lambda(p, [f]^{(n-1)})$ we have

$$\lambda(p, h)=\lambda(p, f) \prod_{j=2}^{r} (1+p^{k-j}).$$

Using the estimation of Raghavan [9, Theorem 1] we have $\lambda(p, h) =O(p^{rk/2})$. We divide into two cases: (i) $\Phi f=0$, and (ii) $\Phi f\neq 0$. In the case (i), by the Ω-result of Rankin [10, Theorem 2] we have $\lambda(p, f) =\Omega(p^{(k-1)/2})$, hence $\lambda(p, f) \prod_{j=2}^{r} (1+p^{k-j})=\Omega(p^{((2r-1)k-r(r+1)+1)/2})$. Hence we have a contradiction if $rk<(2r-1)k-r(r+1)+1$ (i.e., $k>r+2 +(r-1)^{-1}$). This inequality is satisfied, since $r+2+(r-1)^{-1}\leqq n+2 +(n-1)^{-1}$ and k is an even integer larger than $n+2$. In the case (ii), we have $\lambda(p, f)=1+p^{k-1}=\Omega(p^{k-1})$, hence

$$\lambda(p, f) \prod_{j=2}^{r} (1+p^{k-j})=\Omega(p^{(2rk-r(r+1))/2}).$$

Since $k>n+1$, we have $rk<2rk-r(r+1)$ (i.e., $k>r+1$), hence we have a contradiction. Thus we have $\Phi^{n-1}(F)\neq 0$. (We note here that (3) follows from this partial fact by considering $H=F-[f]^{(n-1)}$; if $H\neq 0$ then H is an eigen modular form satisfying $\lambda(p, H)=\lambda(p, [f]^{(n-1)})$ and $\Phi^{n-1}(H)=0$, which is impossible, hence $H=0$.)

Put $g=\Phi^{n-1}(F)$. Then g is an eigen modular form in $M_k(\Gamma_1)$ satisfying $\lambda(p, g)=\lambda(p, f)$. Hence, there exists a non-zero constant $\gamma\in \boldsymbol{C}$ such that $g=\gamma f$ (by the multiplicity one theorem for $M_k(\Gamma_1)$). Put $H=F-\gamma[f]^{(n-1)}$, then $\Phi^{n-1}(H)=0$. If $H\neq 0$, then H is an eigen modular form in $M_k(\Gamma_n)$ satisfying $\lambda(p, H)=\lambda(p, F)=\lambda(p, [f]^{(n-1)})$. This is impossible as shown above. Thus $F=\gamma[f]^{(n-1)}$. Q.E.D.

Remark 2. For $n=2$, Theorem 2(2) is proved alternatively as follows. Assume that $\lambda(F)=\lambda([f])$ as above. Then we have $L(s, F) =L(s, [f])=L(s, f)L(s-k+2, f)$ for the standard L-functions as in [2]. Suppose that $\Phi(F)=0$. Then, from the standard estimation $\lambda(m, F) =O(m^k)$ as $m\to\infty$ (cf. [2, 2.2(3)]), it follows that the Euler product

$$L(s, F)=\prod_{p} (1-(\lambda(p)p^{-s}-(\lambda(p)^2-\lambda(p^2)-p^{2k-4})p^{-2s} +\lambda(p)p^{2k-3-3s}-p^{4k-6-4s}))^{-1}$$

converges absolutely in $\mathrm{Re}(s)>k+1$. Hence $L(s, F)\neq 0$ for $\mathrm{Re}(s) >k+1$. As is well-known, this contradicts to $L(s, F)=L(s, f)L(s-k +2, f)$. Thus $\Phi(F)\neq 0$, and the rest is the same as in the above proof.

§3. Fourier coefficients of Eisenstein series. We prove some rationality (algebraicity, integrality) properties of Fourier coefficients of Eisenstein series treated in §2. We denote by $\mathrm{Aut}(\boldsymbol{C})$ the group of all field-automorphisms of $\boldsymbol{C}$, which acts on $f=\sum_{T\geqq 0} a(T, f)q^T$

$\in M_k(\Gamma_n)$ by $\sigma(f)=\sum_{T\geqq 0}\sigma(a(T,f))q^T \in M_k(\Gamma_n)$ for each $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ (for $n\geqq 0$ and even $k\geqq 0$); see [4].

Theorem 3. *Let f be a modular form in $M_k(\Gamma_1)$. Let $n\geqq 1$ be an integer, and assume that k is an even integer larger than $n+1$ (resp. $n+2$) if $\Phi f \neq 0$ (resp. $\Phi f=0$). Then:*

(1) *For each $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ we have $\sigma([f]^{(n-1)})=[\sigma(f)]^{(n-1)}$.*

(2) *Assume that $f \in M_k(\Gamma_1)_K$ for a subfield K of $\boldsymbol{C}$. Then $[f]^{(n-1)} \in M_k(\Gamma_n)_K$.*

Proof. Let $\{f_1,\cdots,f_r\}$ be an eigen basis (i.e., a basis consisting of eigen modular forms) of $M_k(\Gamma_1)$, and write $f=\sum_{i=1}^r c_i f_i$ with $c_i \in \boldsymbol{C}$. Then, for each $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ we have $\sigma([f]^{(n-1)})=\sum_{i=1}^r \sigma(c_i)\sigma([f_i]^{(n-1)})$ and $[\sigma(f)]^{(n-1)}=\sum_{i=1}^r \sigma(c_i)[\sigma(f_i)]^{(n-1)}$. Hence to prove (1) it is sufficient to prove the case where f is an eigen modular form. Put $F=[f]^{(n-1)}$. Then, for each $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, $\sigma(F)$ is an eigen modular form in $M_k(\Gamma_n)$ (see [4, § 1]) satisfying $\Phi^{n-1}(\sigma(F))=\sigma(f)$. Hence, by Theorem 2(3), we have $\sigma(F)=[\sigma(f)]^{(n-1)}$. This proves (1), hence we have (2) also. Q.E.D.

We say that an eigen modular form $f \in M_k(\Gamma_1)$ is normalized if $a(1,f)=1$.

Theorem 4. *Let f, k, and n be as in the above Theorem 3, and assume that f is a normalized eigen modular form. Then, there exists a non-zero constant $\gamma \in \boldsymbol{Z}(f)$ such that $\gamma[f]^{(n-1)} \in M_k(\Gamma_n)_{\boldsymbol{Z}(f)}$.*

Proof. By Theorem 3(2), $[f]^{(n-1)} \in M_k(\Gamma_n)_{\boldsymbol{Q}(f)}$. On the other hand, by Theorem 2(1) and [4, Theorem 3], there exists a non-zero constant $\gamma_1 \in \boldsymbol{Z}([f]^{(n-1)})$ such that $\gamma_1[f]^{(n-1)} \in M_k(\Gamma_n)_{\boldsymbol{Z}([f]^{(n-1)})}$. (If $n\leqq 2$, we have $\boldsymbol{Z}([f]^{(n-1)})=\boldsymbol{Z}(f)$ by Maass [6], hence we have Theorem 4 for $n\leqq 2$ with $\gamma=\gamma_1$.) Put $\gamma=N(\gamma_1)$ where $N:\boldsymbol{Q}([f]^{(n-1)})\to\boldsymbol{Q}(f)$ denotes the norm map. Then γ is a non-zero constant in $\boldsymbol{Z}(f)$ and $\gamma[f]^{(n-1)}$ belongs to

$$M_k(\Gamma_n)_{\boldsymbol{Q}(f)} \cap M_k(\Gamma_n)_{\boldsymbol{Z}([f]^{(n-1)})}=M_k(\Gamma_n)_{\boldsymbol{Z}(f)}.$$

Hence we have Theorem 4. Q.E.D.

Remark 3. Let f be as in Theorem 4. If $\Phi f\neq 0$, then much more precise results are obtained by Siegel [11] [12] and Maass [7] [8]. The case $\Phi f=0$ is examined in [3] which shows that we can take $\gamma=71^2 11$ (resp. 7) for $[\Delta_{20}]$ (resp. $[\Delta_{12}]$) (these γ being "minimal") and suggests that we can take $\gamma=$the "numerator of $L_2^*(2k-2,f)$" for $[f]$.

Remark 4. As is seen from the proofs of Theorems 2–4, these results would hold in the general situation as in Theorem 1 if a suitable multiplicity one conjecture holds.

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