119. *Higher Order Nonsingular Immersions in Lens Spaces* Mod 3

By Teiichi KOBAYASHI
Department of Mathematics, Faculty of Science, Kochi University

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1. Introduction. H. Suzuki studied in [8] and [9] necessary conditions for the existence of higher order nonsingular immersions of projective spaces in projective spaces by making use of characteristic classes, γ-operations, spin operations, and mod 2 S-relations of stunted real projective spaces.

Let $L^n(q)$ be the $(2n+1)$-dimensional standard lens space mod q. A continuous map $f: L^n(q) \to L^m(q)$ is said to be of degree d ($\in Z_q$) if $f^* x_m = d x_n$, where x_k is the distinguished generator of $H^2(L^k(q); Z_q)$ $(k=m, n)$ and $f^*: H^2(L^m(q); Z_q) \to H^2(L^n(q); Z_q)$ is the homomorphism induced by f. If $m>n$, there is a bijection of the set $[L^n(q), L^m(q)]$ of homotopy classes $[f]$ of continuous maps $f: L^n(q) \to L^m(q)$ onto the group Z_q defined by $[f] \to \deg f$ [5, Lemmas 2.6 and 2.7]. Hence, a continuous map $f: L^n(3) \to L^m(3)$ $(n<m)$ is homotopically non-trivial if and only if $\deg f = \pm 1$. The condition for the existence of homotopically trivial higher order nonsingular immersions of $L^n(q)$ is studied in [6] and [4]. In this paper we are concerned with homotopically non-trivial higher order nonsingular immersions of $L^n(3)$ in $L^m(3)$.

2. Notations and theorems. Let n and k be positive integers. Define an integer A as follows:

$$A = \sum_{j \in \Lambda} \binom{n+j}{j} \binom{n+k-j}{k-j},$$

where $\Lambda = \{j \in Z \mid 0 \leqq j \leqq (k-1)/2$ and $2j \not\equiv k \bmod 3\}$ and $\binom{m}{i} = m!/((m-i)!\,i!)$. For example, $A = n+1$ if $k=1$, $= \binom{n+2}{2}$ if $k=2$, $= (n+1)\binom{n+2}{2}$ if $k=3$, $= \binom{n+4}{4} + (n+1)\binom{n+3}{3}$ if $k=4$. Let $\nu = \nu(2n+1, k)$ denote the dimension $\binom{2n+1+k}{k} - 1$ of the fibre of the kth order tangent bundle $\tau_k(L^n(3))$ of $L^n(3)$.

Theorem 1. *Suppose there exists a homotopically non-trivial kth order nonsingular immersion of $L^n(3)$ in $L^m(3)$ with respect to dissections $\{D_i\}$ on $L^m(3)$.* (i) *If $2m+1 \geqq \nu$, then* $\binom{m+1-A}{j} \equiv 0 \bmod 3$ *for m*

$-[\nu/2]<j\leqq n/2$.

(ii) *If* $0<m-[\nu/2]\leqq n/2$, $\nu+1-2A\not\equiv 0 \bmod 3^{[(n-m-1+\nu/2)/2]}$, *and* ν *is odd, then* $\binom{m+1-A}{m-[\nu/2]}\equiv 0 \bmod 3$. (Here $[x]$ denotes the integral part of an integer x.)

Theorem 2. *Suppose there exists a homotopically non-trivial kth order nonsingular immersion of* $L^n(3)$ *in* $L^m(3)$ *with respect to dissections* $\{D_i\}$ *on* $L^m(3)$. (i) *If* $2m+1\leqq\nu$, *then* $\binom{A-m-1}{j}\equiv 0 \bmod 3$ *for* $[(\nu-1)/2]-m<j\leqq n/2$.

(ii) *If* $0<(\nu-1)/2-m\leqq n/2$, $\nu+1-2A\not\equiv 0 \bmod 3^{[(n+m-\nu/2)/2]}$, *and* ν *is odd, then* $\binom{A-m-1}{(\nu-1)/2-m}\equiv 0 \bmod 3$.

As a consequence of Theorems 1(i) and 2(ii), we have

Corollary 3. *If* $n=3^r$ $(r>1)$, *there is no homotopically non-trivial second order nonsingular immersion of* $L^n(3)$ *in* $L^m(3)$ *for any m such that* $[\nu/2]-[n/2]=n^2+2n+1\leqq m\leqq[\nu/2]+[n/2]-1=n^2+3n-1$, *where* $\nu=\binom{2n+3}{2}-1$.

3. Proofs. For the proofs of theorems we use the following which is proved in [3, Propositions 3.1 and 3.2].

Proposition (4.1). *Let p be an odd prime, and m and n be integers with* $0<m\leqq n/2$. *Assume a positive integer t satisfies*: $\binom{m+t}{m}\not\equiv 0 \bmod p$ *and* $t\not\equiv 0 \bmod p^{[(n-m-1)/(p-1)]}$. *Then* $(m+t)r\eta_n$ *has not independent 2t cross-sections, where* $r\eta_n$ *is the realification of the canonical complex line bundle* η_n *over* $L^n(p)$.

Proof of Theorem 1. (i) Since $2m+1\geqq\nu$, there is the kth order normal bundle $\mu_k(f)$ satisfying

$$\mu_k(f)\oplus\tau_k(L^n(3))=f^!\tau(L^m(3))$$

(cf. [1, Corollary 8.3(a)] or [7, Lemma (2.3)(a)]). By taking the Whitney sum with the trivial line bundle and by making use of the formula due to H. Ôike [6, Theorem 2.8] (cf. also [4, (7.1)]):

$$\tau_k(L^n(3))\oplus 1=Ar\eta_n\oplus(\nu+1-2A),$$

we obtain $\mu_k(f)\oplus Ar\eta_n\oplus(\nu+1-2A)=(m+1)rf^!\eta_m$. By [5, (2.4)], $f^!\eta_m=\eta_n^d$, where $d=\deg f=\pm 1$. Since $r\eta_n^{-1}=r\eta_n$, it follows that

$$(*)\qquad (L+m+1-A)r\eta_n=\mu_k(f)\oplus(2L+\nu+1-2A),$$

for some large integer L such that $L(\eta_n-1)=0$ (cf. [2, Theorem 1]). Since $\dim\mu_k(f)=2m+1-\nu$,

$$p_j(\mu_k(f))=\binom{L+m+1-A}{j}x_n^{2j}=0 \qquad \text{for } j>m-[\nu/2],$$

where p_j denotes the jth Pontrjagin class and x_n is the generator of $H^2(L^n(3);Z_3)$. We may choose L so that $\binom{L+m+1-A}{j}\equiv\binom{m+1-A}{j}$

mod 3. Thus $\binom{m+1-A}{j}\equiv 0$ mod 3 for $m-[\nu/2]<j\leqq n/2$.

(ii) Suppose $\binom{m+1-A}{m-[\nu/2]}\not\equiv 0$ mod 3. Then, by (4.1), assumptions imply that $(L+m+1-A)r\eta_n$ has not independent $2L+\nu+1-2A$ cross-sections. This contradicts the equality $(*)$. Q.E.D.

Proof of Theorem 2. (i) Since $2m+1\leqq\nu$, there is the kth order conormal bundle $\mu'_k(f)$ satisfying

$$\mu'_k(f)\oplus f^!(\tau(L^m(3))=\tau_k(L^n(3))$$

(cf. [1, Corollary 8.3(b)] or [7, Lemma (2.3)(b)]). As in the previous proof, we have

$$(*)' \qquad \mu'_k(f)\oplus(2L+2A-\nu-1)=(L+A-m-1)r\eta_n$$

for some large integer L such that $L(\eta_n-1)=0$. Hence

$$p_j(\mu'_k(f))=\binom{L+A-m-1}{j}x_n^{2j}=0 \qquad \text{for } j>[(\nu-1)/2]-m.$$

We may choose L so that $\binom{L+A-m-1}{j}\equiv\binom{A-m-1}{j}$ mod 3. Therefore $\binom{A-m-1}{j}\equiv 0$ mod 3 for $[(\nu-1)/2]-m<j\leqq n/2$.

(ii) Suppose $\binom{A-m-1}{(\nu-1)/2-m}\not\equiv 0$ mod 3. Then, by (4.1), the assumptions imply that $(L+A-m-1)r\eta_n$ has not independent $2L+2A-\nu-1$ cross-sections. This contradicts the equality $(*)'$. Q.E.D.

Proof of Corollary 3. Suppose there exists a homotopically nontrivial second order nonsingular immersion of $L^n(3)$ in $L^m(3)$ for $m=[\nu/2]+[n/2]-1$. Then we see easily $\binom{m+1-A}{[n/2]}\not\equiv 0$ mod 3. This contradicts Theorem 1(i). Next, suppose there exists a homotopically nontrivial second order nonsingular immersion of $L^n(3)$ in $L^m(3)$ for $m=[\nu/2]-[n/2]$. Then we have $\nu+1-2A\not\equiv 0$ mod $3^{[(n+m-\nu/2)/2]}$ and $\binom{A-m-1}{[n/2]}\equiv(-1)^{[n/2]}\binom{m+[n/2]-A}{[n/2]}\not\equiv 0$ mod 3. This contradicts Theorem 2(ii). $\left(\text{Note that } \nu=\binom{2n+4}{3}-1 \text{ is odd.}\right)$ Q.E.D.

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