113. *Characteristic Indices and Subcharacteristic Indices of Surfaces for Linear Partial Differential Operators*

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Let $P(z,\partial_z)$ be a linear partial differential operator with coefficients holomorphic in Ω, $\Omega\subset C^{n+1}$, and $K=\{\varphi(z)=0\}$ be a nonsingular surface. In the present note we first introduce characteristic indices, subcharacteristic indices and the localization on K of $P(z,\partial_z)$, which represent the relationship between the surface K and $P(z,\partial_z)$. Next we show that they are useful, by considering the equation $P(z,\partial_z)u(z)=f(z)$, where $f(z)$ is holomorphic in $\Omega-K$. The proofs of theorems will be published elsewhere.

§1. **Definitions.** Let C^{n+1} be the $(n+1)$-dimentional complex space. $z=(z_0,z_1,\cdots,z_n)=(z_0,z')$ denotes its point and $\xi=(\xi_0,\xi')$ denotes its dual variable. $\partial_z=(\partial_{z_0},\partial_{z_1},\cdots,\partial_{z_n})=(\partial_{z_0},\partial_{z'})$. For a linear partial differential operator $A(z,\partial_z)$, $A(z,\xi)$ means its total symbol.

Now let us define the localization on K of $P(z,\partial_z)$, characteristic indices σ_i $(1\le i\le p)$ and subcharacteristic indices $\sigma_{p,i}$ $(1\le i\le q)$. We choose the coordinate so that $\varphi(z)=z_0$. Hence $K=\{z_0=0\}$. Let $P(z,\partial_z)$ be a linear partial differential operator of order m in a neighbourhood Ω of $z=0$. Put

$$\begin{cases} P(z,\partial_z)=\sum_{i=0}^{m} P_i(z,\partial_z) \\ P_i(z,\partial_z)=\sum_{l=0}^{i} A_{i,l}(z,\partial_{z'})(\partial_{z_0})^{i-l}, \end{cases} \tag{1.1}$$

where $A_{i,l}(z,\xi')$ is homogeneous in ξ', with degree l. We develop $A_{i,l}(z,\xi')$ with respect to z_0 at $z_0=0$,

$$A_{i,l}(z,\xi')=\sum_{j=0}^{\infty} A_{i,l,j}(z',\xi')(z_0)^j. \tag{1.2}$$

Let us put

$$\begin{cases} d_i=\min\{(l+j)\,;\,A_{i,l,j}(z',\xi')\not\equiv 0\}. \\ j_i=\min\{j\,;\,A_{i,l,j}(z',\xi')\not\equiv 0,\ l+j=d_i\}. \end{cases} \tag{1.3}$$

If $A_{i,l}(z,\xi')\equiv 0$ for all l we put $d_i=j_i=+\infty$. We first give

Definition 1.1. The operator $A_{m,L,J}(z',\partial_{z'})$, where $J=j_m$ and $L+J=d_m$, is called the localization on K of $p(z,\partial_z)$.

Let us define characteristic indices σ_i $(1\le i\le p)$ which were introduced in Ōuchi [4]. Consider the set $A\{(i,d_i)\,;\,0\le i\le m,\ d_i\not=+\infty\}$ in R^2

and the convex hull $\hat{A}$ of A. If the lower convex part of the boundary $\partial\hat{A}$ of $\hat{A}$ consists of one point (m, d_m), we put $\sigma_1=1$. Otherwise it consists of segments $\sum_1(i)$ $(1\leq i\leq l)$ (see Fig. 1.1).

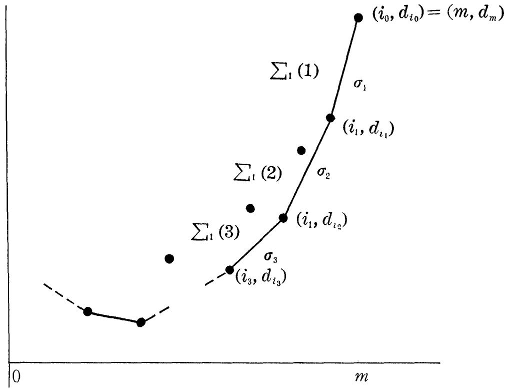

Fig. 1.1

We denote by Δ_1 the set of vertexes of $\bigcup_{i=1}^{l}\sum_1(i)$. We can put $\Delta_1=\{(i_k, d_{i_k});\ k=0, 1, \cdots, l\}$, where $m=i_0>i_1\cdots>i_l\geq 0$. Put

(1.4) $$\sigma_k=\max\{(d_{i_{k-1}}-d_{i_k})/(i_{k-1}-i_k), 1\}.$$

Then there is a $p\in N$ such that $\sigma_1>\sigma_2>\cdots>\sigma_{p-1}>\sigma_p=1$.

Definition 1.2. We call σ_i $(1\leq i\leq p)$ the p-th characteristic index of K for $P(z, \partial_z)$.

Let us define subcharacteristic indices $\sigma_{p,i}$ $(1\leq i\leq q)$. Consider the set $B=\{(i, j_i);\ d_{i_{p-1}}-d_i=i_{p-1}-i,\ 0\leq i\leq i_{p-1}\}$. We also consider the lower convex part of the boundary $\partial\hat{B}$ of the convex hull $\hat{B}$ of B. If $\partial\hat{B}$ consists of one point $(i_{p-1}, j_{i_{p-1}})$, we put $\sigma_{p,1}=1$. Otherwise is consists of segments $\sum_2(i)$ $(1\leq i\leq l')$ (see Fig. 1.2).

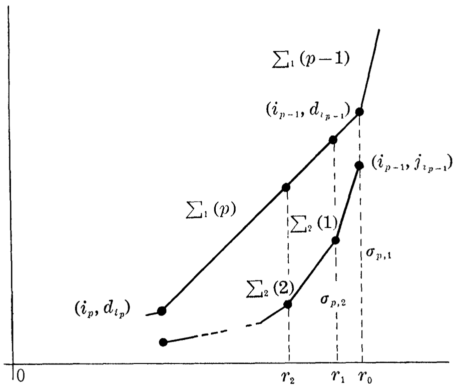

Fig. 1.2

We denote by Δ_2 the set of all vertexes of $\bigcup_{i=1}^{l'}\sum_2(i)$. Put $\Delta_2=\{(r_k, j_{r_k});\ k=0, 1, \cdots, l'\}$, where $i_{p-1}=r_0>r_1>\cdots>r_{l'}\geq 0$, and

(1.5) $$\sigma_{p,k}=\max\{(j_{r_{k-1}}-j_{r_k})/(r_{k-1}-r_k), 1\}.$$

Hence there is a $q \in N$ such that $\sigma_{p,1} > \sigma_{p,2} > \cdots > \sigma_{p,q-1} > \sigma_{p,q} = 1$.

Definition 1.3. $\sigma_{p,i} (1 \leq i \leq q)$ is said to be the i-th subcharacteristic index of K for $P(z, \partial_z)$.

Remark 1.4. K is characteristic, that is, $P_m(0, z', 1, 0, \cdots, 0) \equiv 0$, if and only if $d_m \geq 1$. In particular if $\sigma_1 > 1$ or if $\sigma_1 = 1$ and $\sigma_{1,1} > 1$, then K is characteristic.

We prepare some function spaces:

$\tilde{\mathcal{O}}(\Omega - K)$; the set of all functions holomorphic on the universal covering space of $\Omega - K$,

$\tilde{\mathcal{O}}_\gamma(\Omega - K) = \{f(z) \in \tilde{\mathcal{O}}(\Omega - K)$; for any α, β there are constants $A_{\alpha,\beta}$ and c such that for $\alpha < \arg z_0 < \beta$ and $z \in \Omega - K$ $|f(z)| \leq A_{\alpha,\beta} \exp(c|z_0|^{-\gamma})\}$,

$\tilde{\mathcal{O}}_{(0,\delta)}(\Omega - K) = \{f(z) \in \tilde{\mathcal{O}}(\Omega - K)$; for any α, β there are constants $A_{\alpha,\beta}$ and c such that for $\alpha < \arg z_0 < \beta$ and $z \in \Omega - K$ $|f(z)| \leq A_{\alpha,\beta} \exp(c|\log z_0|^\delta)\}$. We denote $\tilde{\mathcal{O}}_{(0,1)}(\Omega - K)$ by $\tilde{\mathcal{M}}(\Omega - K)$. We have the inclusion $\tilde{\mathcal{M}}(\Omega - K) \subset \tilde{\mathcal{O}}_{(0,\delta)}(\Omega - K) \subset \tilde{\mathcal{O}}_\gamma(\Omega - K)$, where $\gamma > 0$ and $\delta \geq 1$.

§ 2. Theorems. Now let us consider

$$P(z, \partial_z) u(z) = f(z), \tag{2.1}$$

where $f(z) \in \tilde{\mathcal{O}}(\Omega - K)$. For existence of $u(z)$, we have

Theorem I. *Assume that $A_{m,L,J}(0, \xi') \not\equiv 0$, that is, the localization on K is noncharacteristic in some direction at $z' = 0$. Then there is a solution $u(z) \in \tilde{\mathcal{O}}(\Omega_1 - K)$ of (2.1), where Ω_1 is a neighbourhood of $z = 0$ and independent of $f(z)$. Moreover if $\sigma_1 > 1$ and $f(z) \in \tilde{\mathcal{O}}_{\sigma_1 - 1}(\Omega - K)$, $u(z)$ is found in $\tilde{\mathcal{O}}_{\sigma_1 - 1}(\Omega - K)$ and if $\sigma_1 = 1$ and $f(z) \in \tilde{\mathcal{O}}_{(0,\sigma_{1,1})}(\Omega - K)$, $u(z)$ is found in $\tilde{\mathcal{O}}_{(0,\sigma_{1,1})}(\Omega - K)$.*

Corollary II. *Under the assumption of Theorem I, if $f(z) \in \tilde{\mathcal{M}}(\Omega - K)$, there is a solution $u(z)$ of (2.1) in $\tilde{\mathcal{O}}_{\sigma_1 - 1}(\Omega_1 - K)$ ($\tilde{\mathcal{O}}_{(0,\sigma_{1,1})}(\Omega_1 - K)$), if $\sigma_1 > 1$* (resp. $\sigma_1 = 1$).

Remark 2.1. If K is a characteristic surface with constant multiplicity, the equation (2.1) was investigated by Hamada [1], [2], Hamada, Leray and Wagschal [3], Persson [5], Wagschal [6] and others. Their condition, constant multiplicity, is much stronger than ours.

We also have exsitence of null solutions in real domain. We denote by x the real coordinate, $x_i = \operatorname{Re} z_i$.

Theorem III. *Assume that $A_{m,L,J}(0, \xi') \not\equiv 0$ and $L \geq 1$. Then there is a function $u(x)$, which is C^∞ and analytic except $\{x_0 = 0\}$ in neighbourhood U of $x = 0$ such that*

$$\begin{cases} P(x, \partial_x) u(x) = 0 \\ \text{supp. } u(x) \subset \{x_0 \geq 0\} \cap U \\ \text{supp. } u(x) \ni \{x = 0\}. \end{cases} \tag{2.2}$$

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