102. *Analytic Hypo-Ellipticity of a System of Microdifferential Equations with Non-Involutory Characteristics*

By Toshinori ÔAKU
Department of Mathematics, University of Tokyo

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We study the analytic hypo-ellipticity of a system of microdifferential equations whose characteristic variety (in the complex domain) has the form $V=V_1\cup V_2$; here V_1 and V_2 are regular involutory complex submanifolds with non-involutory intersection. We also assume that the system has regular singularities along V (cf. [4]). In particular, the system $(P_1P_2I_m+A)u=0$ satisfies the above conditions if P_1 and P_2 are scalar operators such that the Poisson bracket $\{\sigma(P_1), \sigma(P_2)\}$ does not vanish (where σ denotes the principal symbol), A is an $m\times m$ matrix of operators of lower order, and I_m is the unit matrix of degree m (see Corollary in § 1).

Our result (Theorem in § 1) extends a part of the results of Kashiwara-Kawai-Oshima [3] to more general systems. We believe that our result is new even for single equations (see Example 2). The operator discussed in Corollary is contained in the class discussed by Treves [8] if $\sigma(P_2)$ is the complex conjugate of $\sigma(P_1)$. See also Grušin [1] for a class of single partial differential equations.

§ 1. Statement of the results. Let M be an n-dimensional real analytic manifold and X be its complexification. We denote by $\mathcal{C}_M$ the sheaf on T^*_MX of microfunctions, and by $\mathcal{E}_X$ the sheaf on T^*X of microdifferential operators of finite order. Let $\mathcal{M}$ be a system of microdifferential equations (i.e. a coherent $\mathcal{E}_X$-module) defined on an open subset Ω of T^*X-X. Suppose that the characteristic variety of $\mathcal{M}$ has the form $V=V_1\cup V_2\subset\Omega$. We assume the following conditions (see [4] for notations):

(A.1) V_1 and V_2 are d-codimensional homogeneous regular involutory submanifolds of Ω, and $V_0=V_1\cap V_2$ is non-singular.

(A.2) V_1 and V_2 intersect normally, i.e., $T_pV_1\cap T_pV_2=T_pV_0$ for any $p\in V_0$.

(A.3) $\dim V_1=\dim V_2=\dim V_0+1$.

(A.4) $\operatorname{rank} V_1=\operatorname{rank} V_2=\operatorname{rank} V_0$.

(A.5) $\mathcal{M}$ has regular singularities along V.

Let p_0 be a point of $V_0\cap T^*_MX$. We can find a neighborhood $\Omega'\subset\Omega$ of p_0 and a coherent sub-$\mathcal{E}_V$-module $\mathcal{M}_0$ of $\mathcal{M}|_{\Omega'}$ such that $\mathcal{E}_X\mathcal{M}_0=\mathcal{M}|_{\Omega'}$.

We set $\bar{\mathcal{M}}_0=\mathcal{M}_0/\mathcal{E}(-1)\mathcal{M}_0$. Then we also assume

(A.6) $\bar{\mathcal{M}}_0$ is a locally free $\mathcal{O}_V(0)$-module of rank m.

The polynomial $e_{12}(\lambda, p, \mathcal{M}_0)$ in λ is defined for $p\in V_0\cap\Omega'$ as in [4]. Let $\lambda=\lambda_1, \cdots, \lambda_m$ be the roots of the equation $e_{12}(\lambda, p_0, \mathcal{M}_0)=0$. We assume, in addition, the following two conditions:

(B.1) The generalized Levi form of V_1 (cf. [6, Chapter III]) has at least one negative eigenvalue at p_0.

(B.2) $\lambda_j\notin\{0,1,2,\cdots\}$ for $j=1,\cdots,m$.

Theorem. *Under the assumptions* (A.1)-(A.6) *and* (B.1) *and* (B.2), *the system $\mathcal{M}$ is micro-locally analytic hypo-elliptic at p_0; i.e., we have*

$$\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)_{p_0}=0.$$

Remark. The conclusion of Theorem is also valid if we replace the conditions (B.1) and (B.2) with

(B.1)′ The generalized Levi form of V_2 has at least one negative eigenvalue at p_0.

(B.2)′ $\lambda_j\notin\{-1,-2,-3,\cdots\}$ for $j=1,\cdots,m$.

For a homogeneous holomorphic function f defined in a neighborhood (in T^*X) of p_0, we denote by f^c the complex conjugate of f with respect to T^*_MX; i.e., f^c is the unique holomorphic function such that $f^c=\bar{f}$ holds on T^*_MX.

Corollary. *Let P_1 and P_2 be microdifferential operators of order l_1 and l_2 respectively defined in a neighborhood of $p_0\in T^*_MX-M$. Set $l=l_1+l_2$ and let $A=(A_{ij})$ be an $m\times m$ matrix of microdifferential operators of order at most $l-1$ defined in a neighborhood of p_0. We assume the following conditions*:

$$\sigma(P_1)(p_0)=\sigma(P_2)(p_0)=0, \qquad \{\sigma(P_1), \sigma(P_2)\}(p_0)\neq 0,$$
$$\{\sigma(P_1), \sigma(P_1)^c\}(p_0)<0.$$

We also assume that no eigenvalue of the matrix

$$(\sigma_{l-1}(A_{ij})(p_0)/\{\sigma(P_1), \sigma(P_2)\}(p_0))$$

is a non-negative integer. Then the homomorphism

$$P_1P_2I_m+A: (\mathcal{C}_M)^m_{p_0}\to(\mathcal{C}_M)^m_{p_0}$$

is injective.

Now we give some examples which are contained neither in the class discussed in [3] nor in that discussed in [8]. We set $x=(x_1, x_2)\in\boldsymbol{R}^2$ and $D_j=\partial/\partial x_j$.

Example 1. Set

$$P=(D_1+\sqrt{-1}x_1D_2)(D_1-2\sqrt{-1}x_1D_2)I_m+A_1(x)D_1+A_2(x)D_2+B(x);$$

here A_1, A_2, B are $m\times m$ matrices of real analytic functions defined in an open subset U of $\boldsymbol{R}^2$. Assume that no eigenvalue of the matrix $A_2(0, x_2)$ belongs to $\{3\sqrt{-1}j; j\in\boldsymbol{Z}\}$ for $(0, x_2)\in U$. Then P is analytic hypo-elliptic in U; i.e., if f is a column vector of m hyperfunctions defined in an open subset U' of U such that each component of Pf is real analytic in U', then each component of f is real analytic in U'.

Example 2. Set

$$P=(D_1+\sqrt{-1}x_1D_2)(D_1-\sqrt{-1}(x_1+x_2)D_2)+a_1(x)D_1+a_2(x)D_2+b(x);$$

here $a_1(x)$, $a_2(x)$, and $b(x)$ are real analytic functions defined in a neighborhood of $0=(0,0)\in \boldsymbol{R}^2$. Assume that $a_2(0)\notin\{2\sqrt{-1}j\,;\,j\in \boldsymbol{Z}\}$. Then P is analytic hypo-elliptic at 0; i.e., if f is a hyperfunction defined in a neighborhood of 0 such that Pf is real analytic in a neighborhood of 0, then f is real analytic in a neighborhood of 0.

§ 2. Sketch of the proof. Since the proof of Theorem can be reduced to that of Corollary, we give a sketch of the proof of Corollary. Let $z=(z_1,\cdots,z_n)$ be a local coordinate system of X and $(z,\zeta)=(z_1,\cdots,z_n,\zeta_1,\cdots,\zeta_n)$ be the corresponding local coordinate system of T^*X. We use the notation $D_j=\partial/\partial z_j$. Using methods of [2] and [7], we can find a complex contact transformation φ defined in a neighborhood of p_0 such that

$$\varphi(\{\sigma(P_1)=0\})=\{z_1=0\},\qquad \varphi(\{\sigma(P_2)=0\})=\{\zeta_1=0\},$$

and $\varphi(T^*_MX)=T^*_NX$ in a neighborhood of $\varphi(p_0)=(0,dz_n)$; here $N=\{h(z,\bar z)=0\}$ with a real valued real analytic function h defined in a neighborhood of $0\in X$ such that $h(0)=0$, $d_zh(0)=dz_n$, and that the Levi form of h is positive definite. By a quantized contact transformation Φ associated with φ, we may assume that

$$\Phi^{-1}(P_1P_2I_m+A)=z_1D_1I_m-B;$$

here B is an $m\times m$ matrix of microdifferential operators of order at most 0 commuting with z_1 and D_1, and the real part of each eigenvalue of $\sigma_0(B)(0,dz_n)$ is negative. (See Theorem 1 of [4].) Then it is sufficient to show that the homomorphism

$$z_1D_1I_m-B:\mathcal{H}^1_Z(\mathcal{O}_X)^m_0\to\mathcal{H}^1_Z(\mathcal{O}_X)^m_0$$

is injective; here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X and $Z=\{z\in X\,;\,h(z,\bar z)\geqq 0\}$. The injectivity of this homomorphism can be proved by the method developed in [5].

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