99. *On Hilbert Modular Forms*

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Introduction. In the theory of elliptic modular forms, it is known that every modular form whose Fourier coefficients lie in $\boldsymbol{Z}[1/6]$ is an isobaric polynomial in E_4 and E_6 with coefficients in $\boldsymbol{Z}[1/6]$, where E_4 and E_6 are the normalized Eisenstein series of respective weights four and six.

In this paper, we give an analogous result for Hilbert modular forms for the real quadratic field $K=\boldsymbol{Q}(\sqrt{5})$. Namely, we show that every symmetric Hilbert modular form for K whose Fourier coefficients lie in $\boldsymbol{Z}[1/2]$ can be represented as an isobaric polynomial in certain forms X_2, X_6 and X_{10} with coefficients in $\boldsymbol{Z}[1/2]$.

§ 1. Hilbert modular forms for $\boldsymbol{Q}(\sqrt{5})$. Let $\mathfrak{o}_K$ be the ring of integers in $K=\boldsymbol{Q}(\sqrt{5})$. Let $\boldsymbol{H}$ denote the upper half-plane. Put $\Gamma_K=SL(2,\mathfrak{o}_K)$ and for an element $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Γ_K, we put $\gamma^*=\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$ where the star denotes the conjugation in K.

We let Γ_K operate on $\boldsymbol{H}^2=\boldsymbol{H}\times\boldsymbol{H}$ by:

$$\gamma\cdot(z_1,z_2)=(\gamma z_1,\gamma^* z_2)=\left(\frac{az_1+b}{cz_1+d},\frac{a^*z_2+b^*}{c^*z_2+d^*}\right),\qquad (z_1,z_2)\in\boldsymbol{H}^2.$$

Further, for any $\tau=(z_1,z_2)\in\boldsymbol{H}^2$ and $\nu\in K$, we put

$$N(\nu\tau)=\nu z_1\cdot\nu^* z_2,\qquad tr(\nu\tau)=\nu z_1+\nu^* z_2.$$

A holomorphic function $f(\tau)$ on $\boldsymbol{H}^2$ is called a *symmetric Hilbert modular form of weight* k if it satisfies the following conditions:

(1) For every element $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Γ_K, $f(\tau)$ satisfies a functional equation of the form

$$f(\gamma\cdot\tau)=N(c\tau+d)^k f(\tau);$$

(2) $f((z_1,z_2))=f((z_2,z_1))$.

The set of such functions forms a complex vector space $A_{\boldsymbol{C}}(\Gamma_K)_k$. Any element $f(\tau)$ in $A_{\boldsymbol{C}}(\Gamma_K)_k$ admits a Fourier expansion of the form

$$f(\tau)=\sum_{\substack{\nu\equiv 0 \bmod (1/\sqrt{5}) \\ \nu\gg 0 \text{ or } 0}} a_f(\nu)\exp\left[2\pi i tr(\nu\tau)\right],$$

where the sum extends over all totally positive numbers ν in K satisfying $\nu\equiv 0 \bmod (1/\sqrt{5})$.

For a subring R of $\boldsymbol{C}$, we put

$$A_R(\Gamma_K)_k=\{f\in A_{\boldsymbol{C}}(\Gamma_K)_k \mid a_f(\nu)\in R \text{ for all } \nu\equiv 0\ (1/\sqrt{5}),\ \nu\gg 0 \text{ or } 0\}.$$

Then $A_R(\Gamma_K)_k$ is an R-module and we put $A_R(\Gamma_K)=\oplus_{k\geqq 0} A_R(\Gamma_K)_k$. Any element $f(z)$ in $A_C(SL(2, \boldsymbol{Z}))_k$ has a Fourier expansion:

$$f(z)=\sum_{n=0}^{\infty} a_f(n) \exp(2\pi i n z).$$

For any subring R of $\boldsymbol{C}$, put

$$A_R(SL(2, \boldsymbol{Z}))_k=\{f \in A_C(SL(2, \boldsymbol{Z}))_k \mid a_f(n) \in R \text{ for all } n\geqq 0\}.$$

Next, we consider the ordinary Eisenstein series $G_k(\tau)$ of weight k associated with the modular group Γ_K, which is normalized as the constant term equal to unity (cf. Gundlach [2]). The series $G_k(\tau)$ belongs to $A_C(\Gamma_K)_k$ ($k\geqq 2$) and admits a Fourier expansion:

$$G_k(\tau)=1+\sum_{\substack{\nu\equiv 0 \bmod (1/\sqrt{5}) \\ \nu\gg 0}} b_k(\nu) \exp[2\pi i tr(\nu\tau)],$$

$$b_k(\nu)=\kappa_k \sum_{(\mu)\mid \nu\sqrt{5}} |N(\mu)|^{k-1},$$

$$\kappa_k=(2\pi)^{2k}\cdot\sqrt{5}/[(k-1)!]^2\cdot 5^k\cdot\zeta_K(k),$$

where $\zeta_K(s)$ is the Dedekind zeta function for the field $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$.

Example 1. $\kappa_2=2^3\cdot 3\cdot 5$, $\kappa_4=2^4\cdot 3\cdot 5$, $\kappa_6=2^3\cdot 3^2\cdot 5\cdot 7\cdot 67^{-1}$, $\kappa_{10}=2^3\cdot 3\cdot 5^2\cdot 11\cdot 412751^{-1}$, $\kappa_{12}=2^4\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 691^{-1}\cdot 1150921^{-1}$.

Gundlach [2] constructed a function $\chi_{10}(\tau)$ on $\boldsymbol{H}^2$ as a product of certain theta series on $\boldsymbol{H}^2$ satisfying the following properties: (1) $\chi_{10} \in A_C(\Gamma_K)_{10}$. (2) $\chi_{10}(\tau)$ vanishes on the domain $\Omega=\{\tau=(z_1, z_2) \in \boldsymbol{H}^2 \mid z_1=z_2\}$. The following theorem is proved in [2].

Theorem 1. *If* $f(\tau) \in A_C(\Gamma_K)_k$ *satisfies* $f((z, z))=0$, *then* $f/\chi_{10} \in A_C(\Gamma_K)_{k-10}$.

Now we shall define a linear order among the numbers $\nu \in \boldsymbol{K}$ satisfying $\nu\equiv 0 \bmod (1/\sqrt{5})$ and $\nu\gg 0$ (or $\nu=0$) as follows: First of all, we put

$$\nu=\frac{1}{\sqrt{5}}\,\frac{\alpha+\beta\sqrt{5}}{2}, \quad \alpha, \beta \in \boldsymbol{Z}, \quad \alpha\equiv\beta \bmod 2.$$

Then the conjugation ν^* of ν is given by $\nu^*=(1/\sqrt{5})((-\alpha+\beta\sqrt{5})/2)$ and $tr(\nu)=\beta$.

1. We arrange ν in order of $tr(\nu)$.
2. When the traces are equal, we arrange them in order of α in ν.

We write the numbers ν as $\nu_0, \nu_1, \nu_2, \nu_3, \cdots$ according to this order. We list them for $tr(\nu)\leqq 2$.

trace	$\nu\equiv 0 \bmod (1/\sqrt{5})$, $\nu\gg 0$ or 0
0	$\nu_0=0$
1	$\nu_1=\frac{1}{\sqrt{5}}\frac{-1+\sqrt{5}}{2}$, $\nu_2=\frac{1}{\sqrt{5}}\frac{1+\sqrt{5}}{2}$
2	$\nu_3=\frac{1}{\sqrt{5}}\frac{-4+2\sqrt{5}}{2}$, $\nu_4=\frac{1}{\sqrt{5}}\frac{-2+2\sqrt{5}}{5}$
	$\nu_5=1$, $\nu_6=\frac{1}{\sqrt{5}}\frac{2+2\sqrt{5}}{2}$, $\nu_7=\frac{1}{\sqrt{5}}\frac{4+2\sqrt{5}}{2}$

Now we shall prove a lemma which is required later.

Lemma 1. *Let R be a subring of $\boldsymbol{Q}$. Suppose $f \in A_R(\Gamma_K)_k$, $g \in A_R(\Gamma_K)_{k'}$ $(k \geqq k')$. Furthermore, we assume that the first non zero coefficient of g is invertible in R. If $f = gh$, then $h \in A_R(\Gamma_K)_{k-k'}$.*

Proof. Let $g(\tau) = \sum_{m=n}^{\infty} a_g(\nu_m) \exp[2\pi i tr(\nu\tau)]$, $(a_g(\nu_n) \neq 0)$ and $h(\tau) = \sum_{j=l}^{\infty} a_h(\nu_j) \exp[2\pi i tr(\nu\tau)]$, $(a_h(\nu_l) \neq 0)$. By assumption, $a_g(\nu_n)$ is invertible in R. Suppose $h \notin A_R(\Gamma_K)_{k-k'}$. We assume $a_h(\nu_i)$ is the first coefficient which does not belong to R. Then the coefficient of $\exp[2\pi i tr((\nu_n + \nu_i)\tau)]$ in the expansion of $g(\tau)h(\tau)$ is $a_g(\nu_n)a_h(\nu_i) + \sum a_g(\nu_s)a_h(\nu_t)$, where the sum runs over the numbers ν_s and ν_t $(s > n$ and $t < i)$ such that $\nu_s + \nu_t = \nu_n + \nu_i$. By our assumption, the second sum of above expression must be contained in R. Hence we get $a_g(\nu_n)a_h(\nu_i) \in R$. Since $a_g(\nu_n)$ is invertible in R, we have $a_h(\nu_i) \in R$, which is a contradiction.

§ 2. Hilbert modular forms over $\boldsymbol{Z}[1/2]$. Let R be a subring of $\boldsymbol{C}$. It is known that $f((z,z))$ belongs to $A_R(SL(2,\boldsymbol{Z}))_{2k}$ for any $f(\tau) \in A_R(\Gamma_K)_k$.

Example 2.

$$G_2((z,z)) = E_4(z), \qquad G_2^3((z,z)) - G_6((z,z)) = 2^6 \cdot 3^3 \cdot 5^2 \cdot 67^{-1} \Delta(z),$$

where

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad q = \exp(2\pi i z), \quad \Delta \in A_{\boldsymbol{Z}}(SL(2,\boldsymbol{Z}))_{12}.$$

We define $X_6(\tau) \in A_{\boldsymbol{Q}}(\Gamma_K)_6$ by

$$X_6(\tau) = 2^{-6} \cdot 3^{-3} \cdot 5^{-2} \cdot 67 (G_2^3(\tau) - G_6(\tau)).$$

Lemma 2. $X_6(\tau) \in A_{\boldsymbol{Z}[1/2]}(\Gamma_K)_6$.

Proof. From § 1, we have

$$G_2(\tau) = 1 + 2^3 \cdot 3 \cdot 5 \sum b_2(\nu) \exp[2\pi i tr(\nu\tau)],$$

$$G_6(\tau) = 1 + 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 67^{-1} \sum b_6(\nu) \exp[2\pi i tr(\nu\tau)].$$

Hence we obtain

$$\begin{aligned} G_2^3(\tau) = 1 &+ 2^3 \cdot 3^2 \cdot 5 \sum b_2(\nu) \exp[2\pi i tr(\nu\tau)] \\ &+ 2^6 \cdot 3^3 \cdot 5^2 (\sum b_2(\nu) \exp[2\pi i tr(\nu\tau)])^2 \\ &+ 2^9 \cdot 3^3 \cdot 5^3 (\sum b_2(\nu) \exp[2\pi i tr(\nu\tau)])^3. \end{aligned}$$

By comparing the second terms of $G_2^3(\tau)$ and $G_6(\tau)$, it suffices to prove that

$$2^3 \cdot 3^2 \cdot 5 \cdot 67 b_2(\nu) \equiv 2^3 \cdot 3^2 \cdot 5 \cdot 7 b_6(\nu) \qquad \mathrm{mod}\ 3^3 \cdot 5^2,$$

for all ν. Since $b_k(\nu) = \sum |N(\mu)|^{k-1}$, the above congruence is reduced to the relation $67n \equiv 7n^5 \pmod{3 \cdot 5}$. But we can show this by easy calculation, so we obtain $X_6(\tau) \in A_{\boldsymbol{Z}[1/2]}(\Gamma_K)_6$.

Remark. $X_6(\tau)$ does not belong to $A_{\boldsymbol{Z}}(\Gamma_K)_6$.

The following result is well known.

Lemma 3. *For any $f(z) \in A_{\boldsymbol{Z}[1/2]}(SL(2,\boldsymbol{Z}))_k$, there exists an isobaric polynomial $P(X_1, X_2) \in \boldsymbol{Z}[1/2][X_1, X_2]$ such that $f = P(E_4, \Delta)$, where Δ was defined in Example 2.*

We define $X_{10}(\tau) = \chi_{10}(\tau)$.

Lemma 4. *$X_{10}(\tau) \in A_Z(\Gamma_K)_{10}$ and the coefficient of first term of the expansion in $X_{10}(\tau)$ is power of 2.*

Proof. First we note that χ_{10} is expressed as a product of theta series (cf. [2]) and has the following expression.

$$X_{10}(\tau)=\frac{2^2\cdot 412751}{3^5\cdot 5^5\cdot 7}G_{10}-\frac{2^2\cdot 67\cdot 2293}{3^5\cdot 5^4\cdot 7}G_6G_2^2+\frac{2^4\cdot 4231}{3^4\cdot 5^5}G_2^5$$
$$=2^{12}\exp\,[2\pi i tr(\nu_4\tau)]+\cdots.$$

From this relation, we see that $X_{10}(\tau) \in A_Z(\Gamma_K)_{10}$ and the first coefficient is power of 2.

Now we define $X_2(\tau)=G_2(\tau)$. From Example 1 in § 1, we see $X_2(\tau) \in A_Z(\Gamma_K)_2 \subset A_{Z[1/2]}(\Gamma_K)_2$.

Theorem 2. *For any $f(\tau) \in A_{Z[1/2]}(\Gamma_K)_k$, there exists an isobaric polynomial $F(X_1, X_2, X_3) \in Z[1/2][X_1, X_2, X_3]$ such that $f=F(X_2, X_6, X_{10})$. In other words, the graded $Z[1/2]$-algebra*

$$A_{Z[1/2]}(\Gamma_K)=\bigoplus_{k\geq 2} A_{Z[1/2]}(\Gamma_K)_k$$

is generated by X_2, X_6 and X_{10}.

Proof. If $f(\tau) \in A_{Z[1/2]}(\Gamma_K)_k$, then one verifies, by Lemma 3 that, $f((z, z))=P_0(E_4, \Delta)$ for some $P_0(X_1, X_2) \in Z[1/2][X_1, X_2]$. From Example 2, the function

$$f(\tau)-P_0(G_2(\tau),\ 2^{-6}\cdot 3^{-3}\cdot 5^{-2}\cdot 67(G_2^3(\tau)-G_6(\tau))$$

vanishes on $\Omega=\{\tau=(z_1, z_2) \in H^2 \mid z_1=z_2\}$. Hence, from Theorem 1, the above function is divided by $\chi_{10}=X_{10}$. This implies that

$$f(\tau)=P_0(X_2(\tau),\ X_6(\tau))+f_1(\tau)X_{10}(\tau)$$

for some $f_1(\tau) \in A_Q(\Gamma_K)_{k'}$, $k'+10=k$. If we apply Lemmas 1 and 4 in the case $R=Z[1/2]$, then we get $f_1(\tau) \in A_{Z[1/2]}(\Gamma_K)_{k'}$. We continue to apply a similar argument for $f_1(\tau)$ to obtain

$$f(\tau)=P_0(X_2, X_6)+P_1(X_2, X_6)X_{10}+\cdots+P_j(X_2, X_6)X_{10}{}^j,$$
$$P_i(X_1, X_2) \in Z[1/2][X_1, X_2] \qquad \text{for } 0\leqq i\leqq j.$$

This concludes the proof of Theorem 2.

Remark. J.-I. Igusa determined the generators of the graded ring of Siegel modular forms of degree 2 with rational integral Fourier coefficients (cf. [3]). Some related topics are also found in a recent paper of W. L. Baily, Jr. [1].

References

[1] W. L. Baily, Jr.: A theorem on the finite generation of an algebra of modular forms (1981) (preprint).
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