97. On the Wiener-Schoenberg Theorem for Asymptotic Distribution Functions

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The fundamental theorem of H. Weyl [6] concerning the theory of uniform distribution mod 1 was generalized by E. Hlawka [1] and G. M. Petersen [3] to the case of almost convergence and by M. Tsuji [5] to that of weighted means. Also the study of asymptotic distribution functions mod 1 was initiated by Schoenberg [4]. He obtained the condition under which a sequence should have the asymptotic distribution mod 1.

In this note we shall unify the concepts to show theorems related to the theorem of Schoenberg:

A sequence (x_n) of real numbers has the a.d.f. (mod 1) *if and only if for every positive integer h the limit*

$$\alpha_h=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h x_n}$$

exists and, in addition

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\alpha_h|^2=0.$$

This is also obtained by Wiener [7] in a slightly different form. Now we shall begin with two key definitions:

Let f be a complex-valued continuous function on $(-\infty,+\infty)$ with period 1.

Definition 1. The sequence (x_n) is said to have the (M,λ_n)-asymptotic distribution function mod 1 (abbreviated (M,λ_n)-*a.d.f.* (mod 1)) $g(x)$ if

$$\lim_{n\to\infty}\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k f(x_k)=\int_0^1 f(x)dg(x), \tag{1}$$

where

$$\Lambda_n=\lambda_1+\lambda_2+\cdots+\lambda_n,\quad \lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n\geqq\cdots>0,\qquad \sum_{n=1}^{\infty}\lambda_n=\infty.$$

Definition 2. The sequence (x_n) is said to have the (M,λ_n)-asymptotic well-distribution function mod 1 (abbreviated (M,λ_n)-*a.w.d.f.* (mod 1)) $g(x)$ if

$$\lim_{n\to\infty}\frac{1}{\Lambda_n^k}\sum_{\nu=k+1}^{k+n}\lambda_\nu f(x_\nu)=\int_0^1 f(x)dg(x)\quad \text{uniformly in } k=0,1,2,\cdots, \tag{2}$$

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where

$$\Lambda_n^k = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_{k+n}, \ \lambda_{k+1} \geqq \lambda_{k+2} \geqq \cdots \geqq \lambda_{k+n} \geqq \cdots > 0,$$

$$\sum_{n=1}^{\infty} \lambda_{k+n} = \infty.$$

We also write

$$\Lambda_n^0 = \Lambda_n.$$

Theorem 1. *The sequence* (x_n), $n=1,2,\cdots$, *has the* (M, λ_n)-*a.w.d.f.* (mod 1) $g(x)$ *if and only if for all* $h \in Z$

$$\lim_{n\to\infty} \frac{1}{\Lambda_n^k} \sum_{\nu=k+1}^{k+n} \lambda_\nu e^{2\pi i h x_\nu} = \alpha_h^k \tag{3}$$

exists uniformly in $k=0,1,2,\cdots$, *and*

$$\alpha_h^k = \int_0^1 e^{2\pi i h x} dg(x).$$

Proof. The necessity follows from the fact that the function $e^{2\pi i h x}$ is continuous on $(-\infty, +\infty)$ with period 1. Now assume that (x_n) satisfies (3), and $f(x)$ is a continuous function on $[0,1]$. By Weiestrass' approximation theorem, there exists a complex trigonometric polynomial $P(x)$, that is, a finite linear combination of functions like $e^{2\pi i m x}$ $(m \in Z)$ such that for any positive ε, we have

$$\sup_{0\leqq x\leqq 1} |f(x) - P(x)| < \varepsilon.$$

Thus, for n sufficiently large

$$\left| \frac{1}{\Lambda_n^k} \sum_{\nu=k+1}^{k+n} \lambda_\nu f(x_\nu) - \int_0^1 f(x) dg(x) \right|$$

$$\leqq \left| \frac{1}{\Lambda_n^k} \sum \lambda_\nu f(x_\nu) - \frac{1}{\Lambda_n^k} \sum \lambda_\nu P(x_\nu) \right| + \left| \frac{1}{\Lambda_n^k} \sum \lambda_\nu P(x_\nu) - \int_0^1 P(x) dg(x) \right|$$

$$+ \left| \int_0^1 P(x) dg(x) - \int_0^1 f(x) dg(x) \right|$$

$$\leqq 2\varepsilon + \left| \frac{1}{\Lambda_n^k} \sum \lambda_\nu P(x_\nu) - \int_0^1 P(x) dg(x) \right| < 3\varepsilon,$$

since the last term, as $n \to \infty$, tends to zero uniformly in k by virture of (3). Q.E.D.

Corollary 1. *The sequence* (x_n), $n=1,2,\cdots$, *has the* (M, λ_n)-*a.d.f.* (mod 1) $g(x)$ *if and only if for all* $h \in Z$

$$\lim_{n\to\infty} \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k e^{2\pi i h x_k} = \alpha_h$$

exists and

$$\alpha_h = \int_0^1 e^{2\pi i h x} dg(x).$$

Next we shall state the following extension of Wiener-Schoenberg Theorem, of which the proof runs along the same lines as [2, Chap. 1, Theorem 7.5].

Theorem 2. *The sequence* (x_n) *has a continuous* (M, λ_n)-*a.w.d.f.* (mod 1) *if and only if for every positive integer* h *the limit* (3) *exists* *uniformly in* $k=0,1,2,\cdots$, *and, in addition*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\alpha_h^k|^2=0. \tag{4}$$

Corollary 2. *The sequence* (x_n) *has a continuous* (M,λ_n)*-a.d.f.* (mod 1) *if and only if for every positive integer h*

$$\lim_{n\to\infty}\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k e^{2\pi i h x_k}=\alpha_h$$

exists and, in addition

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|\alpha_h|^2=0.$$

Theorem 3. *Let the sequence* (x_n) *have* (M,λ_n)*-a.w.d.f.* (mod 1) $g(x)$. *Then* $g(x)$ *is absolutely continuous and* $g'(x)\in L^2(0,1)$ *if and only if for all* $h\in Z$

$$\alpha_h^k=\lim_{N\to\infty}\frac{1}{\Lambda_N^k}\sum_{n=k+1}^{k+N}\lambda_n e^{2\pi i h x_n}=\int_0^1 e^{2\pi i h x}dg(x) \tag{5}$$

exists and, in addition

$$\sum_{h=-\infty}^{\infty}|\alpha_h^k|^2<+\infty.$$

Proof. The existence of the limit (5) is necessary. Hence from Parseval's theorem and by the assumption, we have

$$\sum_{h\in Z}|\alpha_h^k|^2=\int_0^1|g'(x)|^2\,dx<+\infty.$$

This proves the necessity. Next we shall show the sufficiency. By Riesz-Fisher theorem, there exists $dg\in L^2(0,1)$ such that

$$\int_0^1 e^{2\pi i n x}dg(x)=\alpha_n^k. \tag{6}$$

Since $dg\in L^2(0,1)\subset L(0,1)$ has a Fourier series that is dominatedly convergent almost everywhere, it follows, after correcting dg on a null set, that

$$dg(x)=\sum_{n\in Z}\alpha_n^k e^{2\pi i n x}\qquad\text{for all } x\in(0,1). \tag{7}$$

From (6), (7) and by Lebesgue's theorem on the derivative of integrals it follows that $g(x)$ is absolutely continuous and $g'\in L^2(0,1)$.

Q.E.D.

Corollary 3. *Let the sequence* (x_n) *have the* (M,λ_n)*-a.d.f.* (mod 1) $g(x)$. *Then* $g(x)$ *is absolutely continuous and* $g'(x)\in L^2(0,1)$ *if and only if for all* $h\in Z$

$$\alpha_h=\lim_{N\to\infty}\frac{1}{\Lambda_N}\sum_{n=1}^{N}\lambda_n e^{2\pi i h x_n}=\int_0^1 e^{2\pi i h x}dg(x)$$

exists and, in addition

$$\sum_{h=-\infty}^{\infty}|\alpha_h|^2<+\infty.$$

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