96. *Periods of Hilbert Modular Surfaces*

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In this note, we announce some results on the Hodge structures attached to Hilbert modular surfaces. Details will appear elsewhere. As an application of these results, we determine the Picard numbers of some Hilbert modular surfaces.

§ 1. **Definitions.** Let D be a prime number such that $D\equiv 1 \bmod 4$. Suppose that $F=\boldsymbol{Q}(\sqrt{D})$ is a real quadratic field with discriminant D whose class number is one. In this case there exists a unit ε in the integer ring $\mathcal{O}_F$ of F such that its norm $N\varepsilon=-1$.

Let $H\times H$ be the product of two copies of the complex upper half plane H. The product $SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$ of the real special linear group $SL_2(\boldsymbol{R})$ of size 2 acts on $H\times H$ factorwise in the usual manner. Composing this action with the natural embedding of $SL_2(\mathcal{O}_F)$ into $SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$ by means of the embeddings of F into $\boldsymbol{R}$, we define the action of $SL_2(\mathcal{O}_F)$ on $H\times H$.

The quotient surface $S=SL_2(\mathcal{O}_F)\backslash(H\times H)$ is a quasiprojective algebraic surface over the complex number field which has a finite number of quotient singularities corresponding to the elliptic fixed points of $SL_2(\mathcal{O}_F)$ in $H\times H$ (cf. Hirzebruch [5]).

§ 2. **The Hodge structure of $H^2(S, \boldsymbol{Q})$.** Let $H^2(S,\boldsymbol{Q})$ be the second rational cohomology group of the surface S. Then, by a theory of Deligne [1], $H^2(S,\boldsymbol{Q})$ has a mixed Hodge structure. Since S has only rational singularities, S is a rational homology manifold, accordingly $W_1H^2(S,\boldsymbol{Q})=0$ by [1]. Therefore $W_2H^2(S,\boldsymbol{Q})$ is a homogeneous Hodge structure of weight 2, which has a polarization

$$\Psi : W_2H^2(S,\boldsymbol{Q})\times W_2H^2(S,\boldsymbol{Q})\longrightarrow \boldsymbol{Q}(-2)$$

induced from the intersection form. We can check that $W_2H^2(S,\boldsymbol{Q})$ is equal to the image of the natural mapping $H^2_c(S,\boldsymbol{Q})\to H^2(S,\boldsymbol{Q})$, where $H^2_c(S,\boldsymbol{Q})$ is the cohomology with compact supports. We may call this $W_2H^2(S,\boldsymbol{Q})$ the Eichler-Shimura cohomology of $SL_2(\mathcal{O}_F)$. On the other hand, $W_2H^2(S,\boldsymbol{Q})$ coincides with a certain cohomology group of square integrable cocycles, which is studied by Harder [3]. We can check that the Chern forms of the automorphy factors

$$\eta_1=\frac{1}{2\pi i}\frac{dz_1\wedge d\bar{z}_1}{y_1^2},\qquad \eta_2=\frac{1}{2\pi i}\frac{dz_2\wedge d\bar{z}_2}{y_2^2}$$

belong to $W_2H^2(S,\boldsymbol{Q})$, where $(z_1,z_2)\in H\times H$, $z_i=x_i+\sqrt{-1}y_i$ $(i=1,2)$.

Let $H^2_{sp}(S, \boldsymbol{Q})$ be the orthogonal complement of $\boldsymbol{Q}\eta_1 \oplus \boldsymbol{Q}\eta_2$ in $W_2H^2(S, \boldsymbol{Q})$ with respect to Ψ. Then $H^2_{sp}(S, \boldsymbol{Q})$ has a homogeneous polarized Hodge structure of weight 2.

Now let us consider the Hecke operators of $SL_2(\mathcal{O}_F)$. Since any Hecke operator gives an algebraic correspondence $\begin{array}{ccc} & T & \\ {}^{p_1}\swarrow & & \searrow{}^{p_2} \\ S & & S \end{array}$ of S such that both p_1 and p_2 are finite flat, we can define the action of T on $H^2(S, \boldsymbol{Q})$ by $p_{1*} \circ p_2^*$. This action of Hecke operators is compatible with the weight filtration and the Hodge filtration. Thus we have the following

Proposition 1 (cf. Hirzebruch [5], Harden [3], and Hida [4]). *Let*

$$H^2_{sp}(S, \boldsymbol{Q}) \otimes \boldsymbol{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

be the Hodge decomposition of $H^2_{sp}(S, \boldsymbol{Q})$. *Then we have natural isomorphism as Hecke ring modules*

$$H^{2,0} \cong \{f(z_1, z_2)dz_1 \wedge dz_2 \mid f \in S_2(SL_2(\mathcal{O}_F))\}$$
$$H^{1,1} \cong \{G^*_\infty(f(z_1, z_2)dz_1 \wedge dz_2) \mid f \in S_2(SL_2(\mathcal{O}_F))\}$$
$$\oplus \{H^*_\infty(f(z_1, z_2)dz_1 \wedge dz_2) \mid f \in S_2(SL_2(\mathcal{O}_F))\}$$
$$H^{0,2} \cong \overline{H}^{2,0} \cong \{G^*_\infty H^*_\infty(f(z_1, z_2)dz_1 \wedge dz_2) \mid f \in S_2(SL_2(\mathcal{O}_F))\}.$$

Here $S_2(SL_2(\mathcal{O}_F))$ *is the space of holomorphic cusp forms of weight* 2 *with respect to* $SL_2(\mathcal{O}_F)$, *and* G_∞ *and* H_∞ *are involutive automorphism of* S *induced from*

$$\tilde{G}_\infty = (z_1, z_2) \in H \times H \longmapsto (\varepsilon z_1, \varepsilon' \bar{z}_2) \in H \times H$$
$$\tilde{H}_\infty = (z_1, z_2) \in H \times H \longmapsto (\varepsilon' \bar{z}_1, \varepsilon z_2) \in H \times H$$

on passing to the quotient, where ε *is a unit satisfying* $\varepsilon > 0$, $\varepsilon' < 0$.

§ 3. Hodge structures attached to primitive forms of weight 2.

If a Hilbert modular cusp form f with respect to $SL_2(\mathcal{O}_F)$ is a common eigenfunction of all Hecke operators, we call it a primitive form. For any primitive form f, we denote by K_f the field over $\boldsymbol{Q}$ generated by the eigenvalues of f. Then K_f is a totally real field in our case. Let f be a primitive form, normalized such that its first Fourier coefficient is 1. Then, for any embedding $\sigma : K_f \hookrightarrow \boldsymbol{C}$, we denote by f^σ the Hilbert modular cusp form obtained from f by applying σ to all the Fourier coefficients of f. f^σ is called a comparison of f (cf. Shimura [7]).

Now let $R_{\boldsymbol{Q}}$ be the $\boldsymbol{Q}$-subalgebra of End $(H^2_{sp}(S, \boldsymbol{Q}))$ generated by the images of Hecke operators. We can show that $R_{\boldsymbol{Q}}$ is a commutative semi-simple algebra. Therefore $R_{\boldsymbol{Q}}$ is isomorphic to a direct sum of fields: $R_{\boldsymbol{Q}} = \oplus_{i=1}^m K_i$. Let M be the finite set consisting normalized primitive forms of weight 2 with respect to $SL_2(\mathcal{O}_F)$. We can write M as a disjoint union $M = \cup M_f$ of subsets M_f, so that each M_f consists of a normalized primitive form and its companions. The cardinality of M_f is the degree $[K_f : \boldsymbol{Q}]$. For each such subset $M_f = \{f^\sigma\}$ ($\sigma : K_f \hookrightarrow \boldsymbol{C}$), we can find a unique subfield K_i of $R_{\boldsymbol{Q}}$ such that $K_i \simeq K_f$ by Multiplicity

One Theorem. Letting e_i be the primitive idempotent of $R_{\boldsymbol{Q}}$ corresponding to the factor K_i we put

$$H^2(M_f, \boldsymbol{Q}) = e_i H^2_{sp}(S, \boldsymbol{Q}).$$

$H^2(M_f, \boldsymbol{Q})$ is a sub-Hodge structure of $H^2_{sp}(S, \boldsymbol{Q})$. Restricting the action of $R_{\boldsymbol{Q}}$, we can define an action of each K_i, accordingly an action

$$\theta_f^* : K_f \hookrightarrow \mathrm{End}\,(H^2(M_f, \boldsymbol{Q}))$$

of K_f via the identification $K_i \rightleftarrows K_f$. Thus we have a direct sum decomposition

$$H^2_{sp}(S, \boldsymbol{Q}) = \bigoplus f \in \Xi H^2(M_f, \boldsymbol{Q}),$$

where Ξ is a subset of M, so that $M = \bigcup_{f \in \Xi} M_f$ (disjoint). Let

$$\Psi_f : H^2(M_f, \boldsymbol{Q}) \times H^2(M_f, \boldsymbol{Q}) \longrightarrow \boldsymbol{Q}(-2)$$

be the polarization obtained by restricting Ψ to $H^2(M_f, \boldsymbol{Q})$. We can show there exists a K_f-bilinear form

$$\psi_f : H^2(M_f, \boldsymbol{Q}) \times H^2(M_f, \boldsymbol{Q}) \longrightarrow K_f$$

so that $\Psi_f = tr_{K_f/\boldsymbol{Q}}(\psi_f)$. We obtain the following

Theorem. (i) *$H^2(M_f, \boldsymbol{Q})$ is a rational polarized Hodge structure of weight 2, on which K_f acts as endomorphisms of Hodge structure via*

$$\theta_f^* : K_f \hookrightarrow \mathrm{End}\,(H^2(M_f, \boldsymbol{Q})).$$

The polarization

$$\Psi_f : H^2(M_f, \boldsymbol{Q}) \times H^2(M_f, \boldsymbol{Q}) \longrightarrow \boldsymbol{Q}(-2)$$

is written as $\Psi_f = tr_{K_f/\boldsymbol{Q}}(\psi_f)$ by a K_f-bilinear non-degenerate symmetric form

$$\psi_f : H^2(M_f, \boldsymbol{Q}) \times H^2(M_f, \boldsymbol{Q}) \longrightarrow K_f.$$

(ii) *For any embedding $\sigma : K_f \hookrightarrow \boldsymbol{R}$ of the totally real number field K_f, the scalar extension $\psi_f \otimes_{K_f, \sigma} \boldsymbol{R}$ has signature $(2, 2)$. For any embedding $\sigma : K_f \hookrightarrow \boldsymbol{C}$, $H^2(M_f, \boldsymbol{Q}) \otimes_{K_f, \sigma} \boldsymbol{C}$ has a Hodge structure of weight 2 with Hodge number $\{h^{2,0} = h^{0,2} = 1,\ h^{1,1} = 2\}$.*

§4. Main theorem A. **Main Theorem A.** *Let f be a primitive form of weight 2 with respect to $SL_2(\mathcal{O}_F)$. Then we can find two isogeny classes $A_f^{(1)} \otimes \boldsymbol{Q}$, $A_f^{(2)} \otimes \boldsymbol{Q}$ of abelian varieties $A_f^{(1)}$, $A_f^{(2)}$ over $\boldsymbol{C}$ of dimension $d = [K_f : \boldsymbol{Q}]$ with endomorphisms*

$$\theta^{(i)*} : K_f \hookrightarrow \mathrm{End}\,(A_f^{(i)}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \qquad (i = 1, 2),$$

such that there exists an isomorphism of Hodge structures

$$H^2(M_f, \boldsymbol{Q}) \xrightarrow{\sim} H^1(A_f^{(1)}, \boldsymbol{Q}) \otimes_{K_f} H^1(A_f^{(2)}, \boldsymbol{Q})$$

compatible with the actions of K_f.

Corollary 1. *Let $H^2_{sp}(S, \boldsymbol{Q})_{\mathrm{alg}}$ be the sub-Hodge structure of $H^2_{sp}(S, \boldsymbol{Q})$ generated by algebraic cycles and let $H^2(M_f, \boldsymbol{Q})_{\mathrm{alg}}$ be its projection to $H^2(M_f, \boldsymbol{Q})$. Then we have an isomorphism of K_f-modules*

$$H^2(M_f, \boldsymbol{Q})_{\mathrm{alg}} \cong \mathrm{Hom}_{\mathcal{O}_f}(A_f^{(1)}, A_f^{(2)}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}.$$

Here $\mathcal{O}_f$ is a sufficiently small order of K_f so that $\theta^{(i)*}(\mathcal{O}_f) \hookrightarrow \mathrm{End}\,(A_f^{(i)})$ for both $i = 1$, and $i = 2$.

Corollary 2. *We have the following equivalences*

(i) $\mathrm{rank}_{K_f} H^2(M_f, \boldsymbol{Q})_{\mathrm{alg}}=0. \Leftrightarrow$ *There is no isogeny between* $A_f^{(1)}$ *and* $A_f^{(2)}$ *compatible with the action of* K_f.

(ii) $\mathrm{rank}_{K_f} H^2(M_f, \boldsymbol{Q})_{\mathrm{alg}}=1. \Leftrightarrow$ *There exists an isogeny* $\phi: A_f^{(1)} \to A_f^{(2)}$ *compatible with the action of* K_f, *but neither* $A_f^{(1)}$ *nor* $A_f^{(2)}$ *is a CM-type abelian variety with a CM-field L satisfying* $[L: K_f]=2$.

(iii) $\mathrm{rank}_{K_f} H^2(M_f, \boldsymbol{Q})_{\mathrm{alg}}=2. \Leftrightarrow$ *Both* $A_f^{(1)}$ *and* $A_f^{(2)}$ *are CM-type abelian varieties with the same CM-field L with* $[L: K_f]=2$ *and there exists an isogeny* $\phi: A_f^{(1)} \to A_f^{(2)}$ *compatible with the action of L.*

§5. Lifting and Hodge structures (Main theorem B). Let $S_2(\Gamma_0(D), (D/*))$ be the space of real Neben type elliptic modular cusp forms of weight 2 with respect to $\Gamma_0(D)$ and with multiplicator $(D/*)$. Then there is a correspondence of primitive forms h of $S_2(\Gamma_0(D), (D/*))$ to primitive form of $S_2(SL_2(\mathcal{O}_F))$ constructed by Naganuma [6] (cf. also Doi-Naganuma [2], Zagier [9]). On the other hand we can attach abelian variety A_h of dimension $[K_h: \boldsymbol{Q}]$ to each primitive form $h \in S_2(\Gamma_0(D), (D/*))$ by the theory of Shimura [7], where K_h is CM-field which is the field of eigenvalues of h. Let $\eta=\begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix}$ be the main involution of $\Gamma_0(D)$ and put $B_h=(1+\eta)A_h$. Then it is known that A_h is isogenous to the product $B_h \times B_h$, and by restriction the maximal totally real subfield $k_h=\{a \in K_h \mid a=a^\rho\}$ acts on B_h as endomorphisms, where ρ is the complex conjugation.

Main Theorem B. *Let h be a primitive form of* $S_2(\Gamma_0(D), (D/*))$, *and let f be a primitive form of* $S_2(SL_2(\mathcal{O}_F))$ *corresponding to h by Naganuma construction* ([6]). *Then* $k_h=K_f$ *and we have isogenies of abelian varieties*

$$\phi_i: B_h \longrightarrow A_f^{(i)} \qquad (i=1,2)$$

compatible with the action of $k_h=K_f$.

Combining this result with a result of Shimura [7], which tells that B_h has no complex multiplication, by Main Theorem A we can show the following

Main Theorem C. *If a primitive form* $f \in S_2(SL_2(\mathcal{O}_F))$ *is obtained from a primitive form h by Naganuma construction, then*

$$\mathrm{rank}_{K_f} H^2(M_f, \boldsymbol{Q})_{\mathrm{alg}}=1.$$

§6. An application. By Theorem C, we can determine completely the Picard numbers of some Hilbert modular surfaces.

Theorem. *Assume that* $D<193$ *or* $D=197, 269, 293, 317$. *Let* $\tilde{S}$ *be a complete smooth model of Hilbert modular surface S, let* $\rho(\tilde{S})$ *be the Picard number of* $\tilde{S}$, i.e. *the rank over* $\boldsymbol{Q}$ *of the subspace of* $H^2(\tilde{S}, \boldsymbol{Q})$ *generated by algebraic cycles, and let* $b_2(\tilde{S})$ *be the second Betti number of* $\tilde{S}$. *Then the birational invariant* $b_2(\tilde{S})-\rho(\tilde{S})$ *is equal to* $3 \cdot p_g(\tilde{S})$, *where* $p_g(\tilde{S})$ *is the geometric genus of* $\tilde{S}$.

§7. Examples on ℓ-adic cohomology groups. The surface S has a canonical model defined over $\boldsymbol{Q}$ ([10]). Let us denote this model over $\boldsymbol{Q}$ by the same symbol S. Now consider the ℓ-adic cohomology group $H^2_{et}(S\times\bar{\boldsymbol{Q}},\boldsymbol{Q}_\ell)$. Since η_1 and η_2 in $W_2H^2_{et}(S\times\bar{\boldsymbol{Q}},\boldsymbol{Q}_\ell)$ are the Chern classes of the line bundles corresponding to the two automorphy factors, we have a natural Gal $(\bar{F}/F)$-module $H^2_{sp}(S\times\bar{\boldsymbol{Q}},\boldsymbol{Q}_\ell)$ so that

$$H^2_{sp}(S\times\bar{\boldsymbol{Q}},\boldsymbol{Q}_\ell)\cong H^2_{sp}(S\times\boldsymbol{C},\boldsymbol{Q})\otimes\boldsymbol{Q}_\ell$$

by the comparison theorem of Artin. Combining the results of [11] with Main Theorem B, we have the following

Theorem. *Assume that* $D=29$, 37 *or* 41. *Then for a sufficiently large algebraic extension* L *of* F, *we have an isomorphism of* Gal $(\bar{\boldsymbol{Q}}/L)$-*modules*

$$H^2_{sp}(S\times\bar{\boldsymbol{Q}},\boldsymbol{Q}_\ell)\cong H^1(B_n\times\bar{\boldsymbol{Q}},\boldsymbol{Q}_\ell)\otimes_{\boldsymbol{Q}_\ell}H^1(B_h\times\bar{\boldsymbol{Q}},\boldsymbol{Q}_\ell).$$

Here B_h *is the elliptic curve attached to the primitive forms* h *and* h *in* $S_2(\Gamma_0(D),(D/*))$ (*note that* $\dim_{C} S_2(\Gamma_0(D),(D/*))=2$).

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