94. A Remark on Certain Stochastic Control Problem

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We consider the following problem of stochastic control in which the system is given by the Ito-type stochastic differential equation: $dX_t=\psi(t,X_t)dt+dB_t$, where ψ is a bounded measurable function, and the cost function to be minimize with respect to ψ is of the form:

$$E\left[\int_s^T L(t,|X_t|)\,dt+h(|X_T|)\right].$$

Our aim is to obtain an explicit form of an optimal control. In addition, as corollary, we can show the existence of solutions of certain partially differential equations of parabolic type with singular drift coefficients.

§ 1. **Representation of optimal control.** Let T be a fixed positive time and assume that $0\leqq s\leqq T$. Consider the following d-dimensional stochastic differential equation:

$$\begin{cases} dX_t=\psi(t,X_t)dt+dB_t, & s\leqq t<T \\ X_s=x, \end{cases} \tag{1.1}$$

where (B_t), $0\leqq t\leqq T$, is a Brownian motion started from 0, ψ is a Borel function from $[0,T]\times R^d$ to R^d such that $\sup_{s,x}|\psi(s,x)|\leqq 1$, and x is a vector (fixed) in R^d. By Ψ we mean the class of such ψ's and any element of Ψ is called *an admissible control.* For any $s\in[0,T)$, $\Psi(s)$ stands for the restriction of Ψ on $[s,T]\times R^d$. Let us note that for all $\psi\in\Psi(s)$ there exists a unique strong solution of Eq. (1.1) in pathwise sense ([4]). By $X_t^{s,x,\psi}$ we mean the solution of Eq. (1.1) associated with $\psi\in\Psi(s)$. The cost function associated with ψ is given by the formula:

$$J(s,x,\psi)=E\left[\int_s^T L(t,X_t^{s,x,\psi})dt+h(X_T^{s,x,\psi})\right] \tag{1.2}$$

where $L(s,x)$ and $h(x)$ are Borel functions from $[0,T]\times R^d$ to R_+ and from R^d to R_+ respectively. An element $\psi^0\in\Psi$ is called *optimal* if ψ^0 satisfies the relation:

$$J(s,x,\psi^0)=\inf_{\psi\in\Psi(s)} J(s,x,\psi),\qquad \text{for all } (s,x). \tag{1.3}$$

Now we need the following notations: $Q^0=(0,T)\times R^d$ and $\bar{Q}^0=[0,T]\times R^d$; $C^j(R^d)$ is the class of functions with continuous partially derivatives of all orders $\leqq j$ on R^d; for $Q\subset R^{d+1}$, $C^{1,2}(Q)$ means the set of $\phi(t,x)$ with $\partial\phi/\partial t$, $\partial\phi/\partial x_i$, $\partial^2\phi/\partial x_i\partial x_j$, $i,j=1,\cdots,d$, continuous on Q; $C_p^{1,2}(Q)$ is the class of $\phi\in C^{1,2}(Q)$ such that $|\phi(t,x)|\leqq c(1+|x|)^k$ for all $(t,x)\in Q$, where c and k are constants not depending upon (t,x), when

ϕ has this property we say that ϕ satisfies the *polynomial growth condition*.

Let $U=\{x \in R^d; |x| \leqq 1\}$ and consider the following Bellman equation:

$$\frac{\partial v}{\partial s}+\frac{1}{2}\Delta v+\min_{a\in U}(a, \nabla v)+L(s, x)=0, \qquad (s, x) \in Q^0, \tag{1.4}$$

with the Cauchy data

$$v(T, x)=h(x), \qquad x \in R^d. \tag{1.5}$$

Suppose the following conditions:

(H.1) a) $L \in C^{0,1}(\bar{Q}^0)$, moreover L and L_x $(=\partial L/\partial x_i, 1 \leqq i \leqq d)$ satisfy the polynomial growth condition, b) $h \in C^2(R^d)$, moreover h and h_x satisfy the polynomial growth condition.

Fleming-Rishel ([1], p. 169–170, Theorems 6.2 and 6.3) proved that if (H.1) holds then Eq. (1.4) with the Cauchy data (1.5) has a unique solution v in $C_p^{1,2}(Q^0)$ with v continuous in $\bar{Q}^0$, and moreover there exists an optimal control in Ψ. In this case we can show that if v is such a unique solution then

$$v(s, x)=\inf_{\psi \in \Psi} J(s, x, \psi), \qquad (s, x) \in Q^0. \tag{1.6}$$

In the following we wish to obtain an explicit form of an optimal control when L and h are particular forms, i.e. we now assume, in addition to (H.1);

(H.2) $L(s, x)=\tilde{L}(s, |x|)$ and $h(x)=\tilde{h}(|x|)$ for all $(s, x) \in Q^0$ where $\tilde{L}$ and $\tilde{h}$ are Borel functions of $[0, T]\times R_+$ to R_+ and R_+ to R_+ respectively. It is not difficult to show that if (H.2) holds then the unique solution v of Eq. (1.4) is also written as $v(s, x)=\tilde{v}(s, |x|)$ for $(s, x) \in \bar{Q}^0$ where $\tilde{v}$ is a Borel function of $[0, T]\times R_+$ to R. In fact, this follows from the fact that all the terms of (1.4) are rotation invariants and from the uniqueness of solutions of Eq. (1.4).

If $|a| \leqq 1$ and $v(s, x)=\tilde{v}(s, |x|)$ then by Schwarz's inequality,

$$|(a, \nabla v)(s, x)| \leqq |\nabla v(s, x)|=|\partial \tilde{v}/\partial r(s, |x|)| \qquad (r=|x|,\ x\neq 0).$$

Define a function ψ^* of $\bar{Q}^0$ to R^d by the formula:

$$\psi^*(s, x)=\begin{cases} -x/|x| & \text{if } \partial \tilde{v}/\partial r(s, |x|) \geqq 0,\ x\neq 0, \\ x/|x| & \text{if } \partial \tilde{v}/\partial r(s, |x|)<0,\ x\neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

then $\psi^* \in \Psi$ and for all $(s, x) \in Q^0$, $(\psi^*, \nabla v)(s, x)=-|\partial \tilde{v}/\partial r(s, |x|)|$. Therefore ψ^* satisfies the following equality:

$$(\psi^*, \nabla v)(s, x)=\min_{a\in U}(a, \nabla v)(s, x), \qquad (s, x) \in Q^0. \tag{1.8}$$

By the Ito-formula and (1.8), it is easily shown that $v(s, x)=J(s, x, \psi^*)$, then by (1.6) this amounts to say that ψ^* is optimal. Thus we have the following theorem.

Theorem 1.1. *Under the hypotheses* (H.1) *and* (H.2), *Eq.* (1.4) *with the Cauchy data* (1.5) *has a unique solution v in $C_p^{1,2}(Q^0)$ with v*

continuous in $\overline{Q}^0$, *such that* $v(s,x)=\tilde{v}(s,|x|)$ *for all* $(s,x)\in\overline{Q}^0$, *where* $\tilde{v}(s,r)$ *is a function of* $C_p^{1,2}((0,T)\times R_+)$. *Moreover* ψ^*, *defined by the formula* (1.7), *is optimal.*

By (1.4) and (1.8), we have the following equation:

$$(1.9)\qquad \begin{cases} \dfrac{\partial v}{\partial s}+\dfrac{1}{2}\Delta v+(\psi^*,\nabla v)+\tilde{L}(s,|x|)=0, & (s,x)\in Q^0,\\ v(T,x)=\tilde{h}(|x|), & x\in R^d. \end{cases}$$

Since $v(s,x)=\tilde{v}(s,|x|)$ from Theorem 1.1 then two equalities,

$$\Delta v(s,x)=\frac{\partial^2\tilde{v}}{\partial r^2}+\frac{d-1}{r}\,\frac{\partial\tilde{v}}{\partial r}(s,|x|)$$

and

$$(\psi^*,\nabla v)(s,x)=-\left(\operatorname{sgn}\frac{\partial\tilde{v}}{\partial r}\right)\frac{\partial\tilde{v}}{\partial r}(s,|x|),$$

imply that $\tilde{v}$ is a solution in $C_p^{1,2}((0,T)\times R_+)$ of the following semilinear partial differential equation of parabolic type:

$$(1.10)\qquad \frac{\partial\xi}{\partial s}+\frac{1}{2}\,\frac{\partial^2\xi}{\partial r^2}+\left(\frac{d-1}{2r}-\operatorname{sgn}\frac{\partial\xi}{\partial r}\right)\frac{\partial\xi}{\partial r}+\tilde{L}(s,r)=0,\qquad 0<s<T,\quad r>0,$$

with the Cauchy data:

$$(1.11)\qquad \xi(T,r)=\tilde{h}(r),\qquad 0<r<\infty,$$

where $\operatorname{sgn}\alpha=1$ if $\alpha\geqq 0$, $=-1$ if $\alpha<0$. Then we have the following.

Corollary 1.2. *Under the same conditions as Theorem* 1.1, $\tilde{v}$ *is a solution in* $C_p^{1,2}((0,T)\times R_+)$ *of Eq.* (1.10) *with the Cauchy data* (1.11).

Remark 1.1. If $\psi^0\in\Psi$ is optimal then $\psi^0=\psi^*$ a.e. on the set $\{(s,x);(\partial\tilde{v}/\partial r)(s,|x|)\neq 0\}$, where a.e. means almost everywhere with respect to R^{d+1} dimensional Lebesgue measure.

§2. Increasing case. In this paragraph we further provide the following condition:

(H.3) If $r\leqq r'$ then $\tilde{L}(s,r)\leqq\tilde{L}(s,r')$ and $\tilde{h}(r)\leqq\tilde{h}(r')$ for all s.

Then we can show that any solution of Eq. (1.10) with the Cauchy data (1.11) is increasing with respect to r for all s. In fact, suppose that $\partial\xi/\partial r<0$ for some point $P^0=(s^0,r^0)$ such that $0<s^0<T$, $0<r^0$, then we can take some neighborhood D of P^0 such that $\partial\xi/\partial r<0$ on D. Eq. (1.10) on D is as follows:

$$(2.1)\qquad \frac{\partial\xi}{\partial s}+\frac{1}{2}\,\frac{\partial\xi^2}{\partial r^2}+\left(\frac{d-1}{2r}+1\right)\frac{\partial\xi}{\partial r}+\tilde{L}(s,r)=0.$$

Differentiating Eq. (2.1) relative to r, we obtain the following:

$$(2.2)\qquad \frac{\partial\xi_r}{\partial s}+\frac{1}{2}\,\frac{\partial^2\xi_r}{\partial r^2}+\left(\frac{d-1}{2r}+1\right)\frac{\partial\xi_r}{\partial r}-\frac{d-1}{2r^2}\,\xi_r+\frac{\partial\tilde{L}}{\partial r}(s,r)=0,$$

where $\xi_r=\partial\xi/\partial r$. From the standard arguments it follows that if $\partial\tilde{L}/\partial r\geqq 0$ then $\xi_r<0$ is impossible. Therefore it holds that $\partial\xi/\partial r\geqq 0$

for all $(s, r) \in [0, T] \times (0, \infty)$. Since $\tilde{v}$ is a solution in $C_p^{1,2}((0, T) \times R_+)$ of Eq. (1.10) by Corollary 1.1, then $\partial \tilde{v}/\partial r \geqq 0$. In this case, ψ^* of (1.7) is equal to ψ', defined by (2.3) below.

(2.3) $$\psi'(s, x) = -x/|x| \text{ if } x \neq 0, \qquad = 0 \text{ if } x = 0.$$

Thus we can summarize as follows.

Theorem 2.1. *Suppose that* (H.1)–(H.3) *hold, then any solution* $\xi(s, r)$ *of Eq.* (1.10) *with the Cauchy data* (1.11) *is increasing with respect to* r *for all* s. *In this case,* ψ', *defined by* (2.3), *is optimal.*

Remark 2.1. 1) Ikeda-Watanabe already proved the fact that ψ' of (2.3) is optimal under the assumptions (H.2) and (H.3) but without (H.1) (see [3, § 2]).

2) In Theorem 2.1, especially if $\tilde{L}(s, r)$ and $\tilde{h}(r)$ are strictly increasing w.r.t. r, then so $\xi(s, r)$ is. This result is deduced by maximum principle for linear partial differential equations of parabolic type (cf. [2, Chap. 2]).

3) In the case of 2), we can conclude by Remark 1.1 that if ψ^0 is optimal then $\psi^0 = \psi'$ a.e. on $[0, T] \times R^d$.

References

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