93. The Stability Theorems for Discrete Dynamical Systems on Two-Dimensional Manifolds

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0. Introduction. One of the basic problems in the theory of dynamical systems is the characterization of stable systems. Let M be a closed (i.e. compact connected without boundary) smooth manifold with a smooth Riemannian metric and $\mathrm{Diff}^r(M)$ denote the space of C^r diffeomorphisms on M with the uniform C^r topology for $r\geq 1$. Let $f\in \mathrm{Diff}^s(M)$ for $s\geq r$. Then f is called *C^r structurally stable* if and only if there is a neighborhood $\mathcal{U}(f)$ of f in $\mathrm{Diff}^r(M)$ such that for any $g\in\mathcal{U}(f)$ there exists a homeomorphism $h: M\to M$ satisfying $gh=hf$. Another important notion of stability is Ω-stability. We denote by $\Omega(f)$ the non-wandering set of f. f is called *C^r Ω-stable* if and only if there is a neighborhood $\mathcal{U}(f)$ of f in $\mathrm{Diff}^r(M)$ such that for any $g\in\mathcal{U}(f)$ there exists a homeomorphism $h: \Omega(f)\to\Omega(g)$ satisfying $gh=hf$ on $\Omega(f)$.

The essential condition to characterize these stabilities is "Axiom A", namely f satisfies *Axiom A* if and only if

(a) $\Omega(f)$ is a hyperbolic set,

(b) $\overline{\mathrm{Per}(f)}=\Omega(f)$,

where $\mathrm{Per}(f)$ denotes the set of all periodic points of f. Recall that a compact f-invariant subset $\Lambda\subset M$ is a *hyperbolic set* if and only if there exist constants $c>0$, $0<\lambda<1$ and a Tf-invariant continuous splitting $TM|\Lambda=E^s\oplus E^u$ such that

$$\|Tf^n|E^s_p\|\leq c\lambda^n,\qquad \|Tf^{-n}|E^u_p\|\leq c\lambda^n,$$

for all $p\in\Lambda$ and non-negative integer n.

In [5], [11] and [4], the following are conjectured.

Structural stability conjecture. *f is C^r structurally stable if and only if f satisfies Axiom A and Strong transversality condition.*

Ω-stability conjecture. *f is C^r Ω-stable if and only if f satisfies Axiom A and No-cycle property.*

For the definitions of Strong transversality condition and No-cycle property, we refer to [5] and [11].

The pupose of this paper is to give an affirmative answer to these conjectures for f of class C^2 in case of $\dim M=2$ and $r=1$.

The sufficiency of Axiom A and Strong transversality condition (resp. No-cycle property) for structural (resp. Ω-) stability has been known for arbitrary dimension [8], [10], [11]. Concerning the necessity, it is known that if Ω-stability implies Axiom A, then both the conjectures are established [4], [9]. In this paper, we investigate a certain class of C^1 diffeomorphisms which contains all C^1 Ω-stable diffeomorphisms, namely we define;

$F(M)=\mathrm{int}_1\{g \in \mathrm{Diff}^1(M)$: all periodic points of g are hyperbolic$\}$, where int_1 means "interior" with respect to C^1 topology of $\mathrm{Diff}^1(M)$.

Our result is the following;

Theorem. *Let* $\dim M=2$ *and* $f \in F(M)$. *If* f *is of class* C^2, *then* $\Omega(f)$ *is a hyperbolic set.*

By the theorem of Kupka-Smale [9], it can be seen that if f is C^1 Ω-stable, then $f \in F(M)$. Furthermore, it is known that Axiom $A(b)$ holds for $f \in F(M)$ (Lemma 3.1 in [2]). Thus, as corollaries of our Theorem, we get;

Structural stability theorem.*) *Let* $\dim M=2$ *and* $f \in \mathrm{Diff}^2(M)$. f *is* C^1 *structurally stable if and only if* f *satisfies Axiom A and Strong transversality condition.*

Ω-stability theorem.*) *Let* $\dim M=2$ *and* $f \in \mathrm{Diff}^2(M)$. f *is* C^1 *Ω-stable if and only if* f *satisfies Axiom A and No-cycle property.*

In this paper, we investigate only C^1 stability because the "C^r Closing lemma" has not been established for $r \geq 2$. For the proof of Axiom $A(b)$ for $f \in F(M)$, we need the "C^1 Closing lemma" [7], and this is again our main tool for the proof of our Theorem.

1. Expansive intervals. From now on, we assume that M is a fixed closed 2-dimensional smooth manifold with a smooth Riemannian metric. For $f \in F(M)$, $p \in \mathrm{Per}(f)$ is hyperbolic and we denote by $E^u_p(f)$ (resp. $E^s_p(f)$) the unstable (resp. stable) subspace of T_pM. For $i=0, 1, 2$, we put

$$\Lambda_i(f)=\text{closure}\,\{p \in \mathrm{Per}(f) : \dim E^s_p(f)=i\}.$$

Note that $\Lambda_0(f)$ and $\Lambda_2(f)$ are the sets of all sources and sinks of f respectively. The following proposition proved by Mañé [1] will play a essential role in studying the precise properties of $f \in F(M)$.

Proposition 1.1. *For* $f \in F(M)$, *there exist* $c>0$ *and* $0<\lambda<1$ *satisfying;*

(i) *there exists a* C^1*-neighborhood* $\mathcal{U}$ *of* f *such that*

$$\|Tg^{\pi(p)}\,|\,E^s_p(g)\| \leq c\lambda^{\pi(p)}, \qquad \|Tg^{-\pi(p)}\,|\,E^u_p(g)\| \leq c\lambda^{\pi(p)}$$

for all $g \in \mathcal{U}$ *and* $p \in \mathrm{Per}(g)$, *where* $\pi(p)$ *denotes the period of* p.

(ii) *there exists a* Tf*-invariant continuous splitting* $TM\,|\,\Lambda_1(f) = E^1 \oplus E^2$ *such that*

*) After I had finished this work, I was informed that in [Chin. Ann. of Math., 1, 9–29 (1980)] S. D. Liao also asserted that he proved the stability conjectures for C^1 diffeomorphisms on 2-manifolds and for C^1 flows with isolated singularities on 3-manifolds. But his method is considerably different from ours.

$$\|Tf^n|E_p^2\|\cdot\|Tf^{-n}|E^1_{f^n(p)}\|\leq c\lambda^n$$

for all $n\in Z_+$ *and* $p\in \Lambda_1(f)$. *Moreover if* $p\in\Lambda_1(f)\cap \mathrm{Per}(f)$, *then* $E^1_p=E^u_p(f)$ *and* $E^2_p=E^s_p(f)$, *where* E^i_p $(i=1,2)$ *denotes the fiber of* E^i *over* p *and* Z_+ *denotes the set of non-negative integers.*

In what follows, we fix a C^2 diffeomorphism f in $F(M)$ and let c, λ be the constants and $TM|\Lambda_1(f)=E^1\oplus E^2$ the splitting both given in Proposition 1.1. We shall concentrate our attention only on E^1 and investigate how the norm of $Tf|E^1$ on a positive orbit of some point $p\in\Lambda_1(f)$ varies. Put $\Lambda=\Lambda_1(f)$ for simplicity.

Definition 1.2. Let $p\in\Lambda$ and $N\in Z_+$. A Z_+-interval $I=[u,v]$ is called a (p,N)-*expansive interval* if and only if I satisfies

(i) length $(I)=v-u\geq N$

(ii) $\|Tf^{v-u}|E^1_{f^u(p)}\|\geq(\lambda^{-1/2})^{v-u}$,

where Z_+-interval means an interval in Z_+.

Such intervals always exist, namely;

Lemma 1.3. *For any* $p\in\Lambda$ *and* $N\in Z_+$, *there exists a* (p,N)-*expansive interval.*

We are interested in where such intervals exist in Z_+ and an answer is given by the next lemma which is the essence of the proof of our Theorem.

Lemma 1.4. *There exists* $N_0\in Z_+$ *such that for* $p\in\Lambda$, $m\in Z_+$ *and integer* $N\geq N_0$ *with*

$$\log\|Tf^m|E^1_p\|\leq(N/2)(\log\lambda),$$

there exists a (p,N)-*expansive interval in* $[0,m]$.

2. Proof of the theorem. From Lemma 3.1 in [2], we have $\Omega(f)=\Lambda_0(f)\cup\Lambda_1(f)\cup\Lambda_2(f)$. In [6], Pliss proved that $\Lambda_0(f)$ and $\Lambda_2(f)$ are finite sets and consequently they are hyperbolic sets. Therefore we have only to show that $\Lambda=\Lambda_1(f)$ is a hyperbolic set, namely;

(1) there exist $d>0$ and $0<\mu<1$ such that

(i) $\|Tf^{-n}|E^1_p\|\leq d\mu^n$

(ii) $\|Tf^n|E^2_p\|\leq d\mu^n$

for all $n\in Z_+$ and $p\in\Lambda$.

We shall only prove (i). The same argument as in (i) with f^{-1} on E^2 gives (ii). Suppose (i) does not hold, then by an argument in [1], we have that

(2) there exists a $p_*\in\Lambda$ such that $\|Tf^n|E^1_{p*}\|\leq 1$ for all $n\in Z_+$.

We fix a positive integer $N\geq N_0$. From Lemma 1.3, there exists a (p_*,N)-expansive interval. Let $I=[a,b]$ be the minimum element of the set of all (p_*,N)-expansive intervals, where the order for intervals is given lexicographically, that is; for $I_1=[m_1,n_1]$ and $I_2=[m_2,n_2]$, $I_1\leq I_2$ iff $m_1<m_2$ or, $m_1=m_2$ and $n_1\leq n_2$. From the definition of expansive intervals, we have

$$\|Tf^b|E^1_{p_*}\|=\|Tf^{b-a}|E^1_{f^a(p_*)}\|\cdot\|Tf^a|E^1_{p_*}\|\geq(\lambda^{-1/2})^N\cdot\|Tf^a|E^1_{p_*}\|.$$

From this and (2), we get

$$\log\|Tf^a|E^1_{p_*}\|\leq(N/2)(\log\lambda).$$

Then by Lemma 1.4, it follows that there exists a (p_*, N)-expansive interval in $[0, a]$. This contradicts the fact that $I=[a, b]$ is the minimum among all (p_*, N)-expansive intervals. This completes the proof.

3. Main lemma. Lemma 1.3 is proved by using the technique of the "C^1 Closing lemma" [7] and Proposition 1.1 (i) with rather rough estimates. While the basic idea of the proof of Lemma 1.4 is the same as that of Lemma 1.3, we need more delicate estimates. By precise computations with certain C^2 invariant coordinates, we can see that for all $K>0$ and $\theta>3$ there exists an $n_0\in Z_+$ such that for $p\in\Lambda$, $m\in Z_+$ and integer $N\geq n_0$ with

(i) $\|Tf^m|E^1_p\|\leq 1$

(ii) $m\geq K\cdot N^\theta$,

there exists a (p, N)-expansive interval in $[0, m]$. Thus, if there exist constants $K>0$ and $\theta>3$ such that the condition

$$\log\|Tf^m|E^1_p\|\leq(N/2)(\log\lambda)$$

implies $m\geq K\cdot N^\theta$ for all $N\in Z_+$, then Lemma 1.4 is established. This is guaranteed by the following

Lemma 3.1. *There exists a constant $K>0$ such that if for $l>0$ and $m\in Z_+$, $\log\|Tf^m|E^1_p\|\leq -l$ holds at some $p\in\Lambda$, then $m\geq K\cdot l^4$.*

This lemma is proved by applying the following Main lemma inductively.

Main lemma. *There exist $N^*\in Z_+$ and $G>0$ with the following property: Let $p\in\Lambda$. If for integers $N\geq N^*$ and $m>0$*

$$\log\|Tf^m|E^1_p\|\leq(N^{10/9}/2)(\log\lambda)$$

holds, then there exist integers $0\leq m_1<m$, $0<\nu\leq[GN]$, and ν disjoint Z_+-intervals $\{I_i=[u_i, v_i]\}_{1\leq i\leq\nu}$ in $[0, m-m_1]$ with the properties;

(i) $\|Tf^n|E^1_{f^{m_1}(p)}\|\leq 1$ *for all* $0\leq n\leq m-m_1$

(ii) $\sum_{i=1}^{\nu}(v_i-u_i)>[GN]\cdot N$

(iii) $\|Tf^{v_i-u_i}|E^1_{f^{u_i+m_1}(p)}\|\geq(\lambda^{-1})^{v_i-u_i}$ *for all* $1\leq i\leq\nu$.

In the proof of this Main lemma, we use the technique of the "C^1 Closing lemma" for *closing up* and *cutting off* the suborbits of the positive orbit of given $p\in\Lambda$.

The details will be published elsewhere.

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