91. *Singular Cauchy Problems for a Class of Weakly Hyperbolic Differential Operators*

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In these notes singular Cauchy problems of Hamada's type are studied in the category of holomorphic functions and hyperfunctions for a class of hyperbolic differential operators with non-involutive multiple characteristics. Integral representations of their solutions are given.

1. Introduction. Let $P(t,x,D_t,D_x)$ be a differential operator of order m of the form

$$P(t,x,D_t,D_x)=D_t^m+\sum_{i=1}^m A_i(t,x,D_x)D_t^{m-i},$$

where $D_t=(1/\sqrt{-1})(\partial/\partial t)$, $D_x=(1/\sqrt{-1})(\partial/\partial x)$ and $A_i(t,x,D_x)$ is a differential operator at most of order i, not containing D_t, whose coefficients are holomorphic functions defined in a neighborhood of $(t,x)=(0,0)$ in $C\times C^n$.

We assume the following conditions:

(A-1) (Degeneracy of characteristic roots). There exists a non-negative integer q such that the principal symbol $P_m(t,x,\tau,\xi)$ of $P(t,x,D_t,D_x)$ is expressed in the form

$$P_m(t,x,\tau,\xi)=\prod_{j=1}^m(\tau-t^q\lambda_j(\xi)),$$

where $\lambda_j(\xi)$ $(1\leq j\leq m)$ are holomorphic functions defined in a conic open neighborhood Ω_0 of $\xi_0=(1,0,\cdots,0)$ in C^n-0 and homogeneous of degree 1 such that

$$\lambda_j(\xi)\neq\lambda_k(\xi), \qquad \text{if } j\neq k \text{ and } \xi\in\Omega_0.$$

(A-2) (Hyperbolicity). $\lambda_j(\xi)$ $(1\leq j\leq m)$ are real if ξ is real.

(A-3) (Levi condition). Let $A_{i,j}(t,x,\xi)$ be the homogeneous part of $A_i(t,x,\xi)$ of degree j with respect to ξ and let

$$A_{i,j}(t,x,\xi)=\sum_{k=0}^\infty t^k A_{i,j,k}(x,\xi)$$

be the Taylor expansion of $A_{i,j}(t,x,\xi)$ with respect to t. Then

$$A_{i,j,k}(x,\xi)=0, \qquad \text{if } k<(q+1)j-i.$$

Alinhac [1], Amano [2], Amano-Nakamura [13], Nakamura-Uryu [6], Nakane [7], Taniguchi-Tozaki [10] and Yoshikawa [12] studied the Cauchy problem for weakly hyperbolic operators of the above type, and constructed parametrices, using a type of ordinary differential operators with polynomial coefficients which determine the principal parts of parametrices. (Nakamura-Uryu [6] and Amano-Nakamura [13] studied a more general case.) All of these authors, except Nakane [7], treated these subjects in the category of C^∞ functions.

We deal with the singular Cauchy problem of Hamada's type

$$(\mathrm{CP})_i \quad \begin{cases} P(t,x,D_t,D_x)u_i(t,x,y,\xi)=0, \\ D_t^j u_i|_{t=0}=\delta_{j,i}(\langle x-y,\ \xi\rangle+\sqrt{-1}\,0)^{-n} \qquad \text{for } 0\le j\le m-1, \end{cases}$$

and its version in the complex domains

$$(\mathrm{CP})_i^{C} \quad \begin{cases} P(t,x,D_t,D_x)u_i(t,x,y,\xi)=0, \\ D_t^j u_i|_{t=0}=\delta_{j,i}\langle x-y,\ \xi\rangle^{-n} \qquad \text{for } 0\le j\le m-1, \end{cases}$$

for $0\le i\le m-1$, where $\delta_{j,i}$ is Kronecker's delta, and show that the solution u_i is obtained as an infinite series of Radon integrals (see (1.2)). Parametrices are obtained as integrals

$$\int_{|\xi|=1} u_i(t,x,y,\xi)\omega(\xi) \qquad (0\le i\le m-1),$$

where

$$\omega(\xi)=\sum_{j=1}^{n}(-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n.$$

Our construction of solutions is similar to those of Yoshikawa [12] and Nakamura-Uryu [6], but we need much more delicate estimations. Our Radon integrals are modifications of those studied by Kataoka [4] and Aoki [3].

Before stating our main theorems we introduce the notations

$\psi_j(t,\xi)=(q+1)^{-1}\lambda_j(\xi)t^{q+1}$,

$\varphi_j(t,x,y,\xi)=\langle x-y,\xi\rangle+\psi_j(t,\xi)$,

$r_j(t,\xi)=\max_{1\le k\le m}|\psi_j(t,\xi_1^{-1}\xi)-\psi_k(t,\xi_1^{-1}\xi)|$,

$d(t,\xi_1)=(|t|^{(q+1)}+|\xi_1|^{-1})^{1/(q+1)}$,

$X=\{x\in \boldsymbol{C}^n;\ |x|<a\}$,

$\Omega=\{\xi=(\xi_1,\xi')\in \boldsymbol{C}^n-0;\ |\xi'|<b\,|\xi_1|,\ |\arg(\xi_1)|<b\}$,

$S_\sigma=\{t\in \boldsymbol{C}-0;\ |\arg(\sigma t)|<(2q+2)^{-1}\pi-\varepsilon\}$,

$Z_\sigma=S_\sigma\times X\times\Omega, \qquad Z=\boldsymbol{C}\times X\times\Omega$,

$D_0(r)=\{p\in \boldsymbol{C};\ \mathrm{Im}\,(p)>r\}$,

$D_1(d,r,R)=\bigcup_{-b<\theta<b}\{p\in \boldsymbol{C};\ \mathrm{Im}\,(pe^{\sqrt{-1}\theta})>r,\ de^{R(|p|+r)}<1\}$,

for positive constants a,b,d,r,R and $\sigma=\pm1$, $1\le j\le m$.

Under Assumptions (A-1)–(A-3) we have

Theorem 1. *For any sufficiently small positive constant ε, there exist positive constants a,b,h and holomorphic functions $u_{\sigma,i,j}^{(\nu)}$ ($\sigma=\pm1$, $0\le i\le m-1$, $1\le j\le m$, $\nu\ge0$) defined in Z such that the following hold:*

(i) *A solution u_i of $(\mathrm{CP})_i^C$ is obtained in the form*

$$u_i(t,x,y,\xi)=\sum_{j=1}^{m}u_{\sigma,i,j,R}(t,x,\xi;\varphi_j(t,x,y,\xi))+h_{\sigma,i,R}(t,x,y,\xi), \tag{1.1}$$

for $\sigma=\pm1$, where $u_{\sigma,i,j,R}$ is given by

(1.2) $u_{\sigma,i,j,R}(t,x,\xi;p)$

$$=\sum_{\nu=0}^{\infty}\int_{(\nu+1)R}^{\infty}\exp(\sqrt{-1}\xi_1^{-1}p\rho)\xi_1^{-n}u_{\sigma,i,j}^{(\nu)}(t,x,\rho\xi_1^{-1}\xi)\rho^{n-1}d\rho,$$

for any positive constant $R>h^{q+1}$, *and* $h_{\sigma,i,R}$ *is a holomorphic function defined in a neighborhood of* $(t,x,y,\xi)=(0,0,0,\xi_0)$ *and homogeneous of degree* $(-n)$ *with respect to* ξ.

(ii) $u_{\sigma,i,j}^{(\nu)}$ *satisfies* (2.1)–(2.4).

(iii) *The series* (1.2) *converges uniformly in every compact subset of the domain*

$$\{(t,x,\xi,p)\in Z_\sigma\times \boldsymbol{C};\, d(t,R)h<1,\ \xi_1^{-1}p\in D_0(0)\}$$
$$\cup\bigcap_{j=1}^{m}\{(t,x,\xi,p)\in Z\times\boldsymbol{C};\, d(t,R)h<1,\ \xi_1^{-1}p\in D_0(r_j(t,\xi))\},$$

and hence it is a holomorphic function which is defined in this domain and homogeneous of degree $(-n)$ *with respect to* (ξ,p).

(iv) *By deforming the integration path of the* ν*-th term of* (1.2) *into*

$C_{\nu,R,\theta}=\{(\nu+1)R\exp(\sqrt{-1}s\theta);\,0\le s\le 1\}\cup\{(\nu+1)Rs\exp(\sqrt{-1}\theta);\,1\le s\}$ $(|\theta|<b)$, $u_{\sigma,i,j,R}$ *is continued to a holomorphic function defined in the domain*

$$\{(t,x,\xi,p)\in Z_\sigma\times\boldsymbol{C};\,\xi_1^{-1}p\in D_1(d(t,R)h,0,R)\}$$
$$\cup\bigcap_{j=1}^{m}\{(t,x,\xi,p)\in Z\times\boldsymbol{C};\,\xi_1^{-1}p\in D_1(d(t,R)h,r_j(t,\xi),R)\}.$$

Theorem 2. *The solution of* $(\mathrm{CP})_i$ *is given by the "boundary value hyperfunction" of* (1.1), *namely,*

(1.3) $u_i(t,x,y,\xi)$

$$=\sum_{j=1}^{m}u_{\sigma,i,j,R}(t,x,\xi;\varphi_j(t,x,y,\xi)+\sqrt{-1}\,0)+h_{\sigma,i,R}(t,x,y,\xi).$$

The singularity support and the singularity spectrum of u_i, *if we regard* ξ *as a parameter, are estimated as follows:*

$$\text{(1.4)}\qquad \text{sing. supp.}\,(u_i)\subset\bigcup_{j=1}^{m}\{\varphi_j=0\},$$

$$\text{S.S.}\,(u_i)\subset\bigcup_{j=1}^{m}\{(t,x,y;\sqrt{-1}d\varphi_j(t,x,y,\xi)\infty);\,\varphi_j=0\}.$$

(As for the terminologies of hyperfunctions and singularity spectra, we refer to Sato-Kawai-Kashiwara [9].)

Remark 1. As Amano [2] and Amano-Nakamura [13] pointed out, our method for the construction of solutions will be effective in the analysis of the "branching of singularities" at multiple characteristic points. Alinhac [1], Nakane [7] and Taniguchi-Tozaki [10] carried out the analysis in the case $m=2$, using special functions of the hypergeometric or confluent hypergeometric type.

Remark 2. (1.4) implies that the singularities of solutions are concentrated on the union of bicharacteristic strips associated with $\tau-t^q\lambda_j$ $(1\le j\le m)$ which pass $(t,x,y,\xi)=(0,0,0,\xi_0)$. This result is entirely different from those in the case of involutive multiple characteristics. (See, for example, Kawai-Nakamura [5].)

2. Outline of the proof. We choose $u^{(\nu)}_{\sigma,i,j}(t,x,\xi)$ to be "semi-homogeneous" of degree $(-i-\nu)/(q+1)$, namely, for $c\in \boldsymbol{C}-0$,

$$u^{(\nu)}_{\sigma,i,j}(c^{-1}t,x,c^{q+1}\xi)=c^{-i-\nu}u^{(\nu)}_{\sigma,i,j}(t,x,\xi) \tag{2.1}$$

holds. Then, at least formally, we can reduce the problem to the "transport equation"

$$\begin{aligned}(2.2)\qquad &\Big(D_t^m+\sum_{k=1}^m A_k^{(0)}(t,x,\xi)D_t^{m-k}\Big)(\exp(\sqrt{-1}\psi_j(t,\xi))u^{(\nu)}_{\sigma,i,j})\\ &=-\sum_{k=1}^m \sum_{\substack{\kappa,\lambda,\alpha\geq 0,\\ \nu=\kappa+\lambda+(q+1)|\alpha|,\\ \nu\neq\lambda}} \frac{1}{\alpha!}(\partial_\xi^\alpha A_k^{(\kappa)})D_x^\alpha D_t^{m-k}\\ &\qquad\times(\exp(\sqrt{-1}\psi_j(t,\xi))u^{(\lambda)}_{\sigma,i,j}),\end{aligned}$$

with the "initial condition"

$$\sum_{j=1}^m D_t^k(\exp(\sqrt{-1}\psi_j)u^{(\nu)}_{\sigma,i,j})|_{t=0}=\delta_{k,i}\delta_{\nu,0}(\sqrt{-1})^{-n}/(n-1)!, \tag{2.3}$$

for $0\leq k\leq m-1$, $1\leq k\leq m$, $\nu\geq 0$, where $\partial_\xi^\alpha=(\partial/\partial\xi)^\alpha$, and $A^{(\nu)}_{\sigma,i,j}(t,x,\xi)$ is defined by

$$A_i^{(\nu)}(t,x,\xi)=\sum_{\substack{k\geq 0,\,0\leq j\leq i\\ \nu=k-(q+1)j+i}} t^k A_{i,j,k}(x,\xi).$$

Actually we can show that the proof is reduced to the construction of solutions of (2.2) and (2.3) which satisfy the "growth condition"

$$\begin{aligned}(2.4)\qquad &|D_t^k u^{(\nu)}_{\sigma,i,j}|\leq Ch^\nu|\xi_1|^{k-(q+1)^{-1}i}d(t,\xi_1)^{kq}d(t,(\nu+1)^{-1}\xi_1)^\nu\\ &\times\begin{cases} d(t\xi_1^{1/(q+1)},1)^{\mu_j(x,\xi)}, & \text{if } (t,x,\xi)\in Z_\sigma,\\ d(t\xi_1^{1/(q+1)},1)^{M(x,\xi)}e^{r_j(t,\xi)|\xi_1|}, & \text{if } (t,x,\xi)\in Z,\end{cases}\end{aligned}$$

for $0\leq k\leq m$ and $\nu\geq 0$, where we set

$$\begin{aligned}\pi_j(x,\xi)=-\sum_{i=1}^m\{&(q/2)(m-i+1)(m-i)A_{i-1,i-1,q(i-1)}(x,\xi)\\ &+\sqrt{-1}A_{i,i-1,q(i-1)-1}(x,\xi)\}\\ &\times\lambda_j(\xi)^{m-i}\prod_{k=1,k\neq j}^m(\lambda_j(\xi)-\lambda_k(\xi))^{-1},\end{aligned}$$

$$\mu_j(x,\xi)=\mathrm{Re}\,(\pi_j(x,\xi)),\qquad M(x,\xi)=\max_{1\leq j\leq m}\mu_j(x,\xi).$$

Since $\psi_j(t,\xi)$ and $A_i^{(\nu)}(t,x,\xi)$ are semi-homogeneous of degree 0 and $(i-\nu)/(q+1)$ respectively, (2.2)–(2.4) are compatible with the semi-homogeneity. Namely, if $u^{(\nu)}_{\sigma,i,j}(t,x,\xi)$ ($\sigma=\pm 1$, $1\leq j\leq m$, $0\leq i\leq m-1$, $\nu\geq 0$) satisfy (2.2)–(2.4) with the constraint $\xi_1=1$ (which we abbreviate to $(2.2)|_{\xi_1=1}$, $-(2.4)|_{\xi_1=1}$), and if we set

$$u^{(\nu)}_{\sigma,i,j}(t,x,\xi)=\xi_1^{(-i-\nu)/(q+1)}u^{(\nu)}_{\sigma,i,j}(t\xi_\nu^{1/(q+1)},x,\xi_1^{-1}\xi'),$$

then they are solutions of (2.2)–(2.4). Thus essentially we have only to consider the case $\xi_1=1$.

We construct $u^{(\nu)}_{\sigma,i,j}$ by the induction on ν.

For $\nu=0$, $(2.2)|_{\xi_1=1}$ is a homogeneous ordinary differential equation with polynomial coefficients with respect to t, and has an irregular singular point of Poincaré's rank $(q+1)$ at $t=\infty$. It has formal solutions of the form

$$v_j = \hat{v}_j t^{\kappa_j(x,1,\xi')} \exp(\sqrt{-1}\psi_j(t,1,\xi')) \qquad \text{for } 1 \leq j \leq m,$$

where $\hat{v}_j$ is a formal power series in $t^{-(q+1)}$ whose coefficients are holomorphic functions in (x, ξ'). By a version of the "asymptotic existence theorem" (Wasow [11, Theorem 12.3]), there exists, for each j, a holomorphic solution (not formal) of $(2.2)|_{\xi_1=1}$ whose asymptotic expansion in the sector S_σ coincides with v_j. Using these solutions we get solutions of (2.2)–(2.4) for $\nu=0$.

For $\nu \neq 0$, we construct solutions by the "method of the variation of constants", using the same integration paths as those which appeared in Nishimoto [8]. Through some delicate estimations we can show that these solutions actually satisfy the former half of (2.4).

To show the latter half of (2.4), we choose a suitable family of finitely many sectors which cover the whole plane $\boldsymbol{C}$. Then we repeat similar arguments to express $u^{(\nu)}_{\sigma,i,k}$, for each sector S of this family, in such a manner that

$$u^{(\nu)}_{\sigma,i,k} = \sum_{j=1}^{m} \exp(\sqrt{-1}(\psi_j - \psi_k)) u^{(\nu)}_{\sigma,i,k;S,j}$$

holds, where $u^{(\nu)}_{\sigma,i,k;S,j}$ satisfies the transport equations and the former half of (2.4) in S, instead of S_σ. This implies the latter half of (2.4).

The last argument is indispensable, because "Stokes' phenomena" (Wasow [11, § 15]) may occur and the factor $\exp(\sqrt{-1}(\psi_j - \psi_k))(j \neq k)$ may break the validity of the former half of (2.4) in the sector $S\ (\neq S_\sigma)$. This is the reason why we need the latter half of (2.4). This relates to the "branching of singularities". (See Remark 1.)

The detailed proof will appear elsewhere.

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