# 90. *Vertex Operators and $\tau$ Functions*

## *Transformation Groups for Soliton Equations. II*

By Etsuro DATE,*) Masaki KASHIWARA,**) and Tetsuji MIWA**)

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1981)

It was E. Galois who noticed the importance of the transformation groups of algebraic equations. This idea has been powerful in several fields. The theory of Picard-Vessiot, for example, is to study linear ordinary differential equations by the aid of Lie groups acting on them.

The purpose of this article is to add another example to the philosophy of E. Galois by studying soliton-type equations such as the Korteveg de Vries equation (KdV equation), the Boussinesq equation, the Kadomtsev-Petviashvili equation (KP equation), etc. through their transformation groups. In these cases, the groups are not finite-dimensional any more, but infinite-dimensional Lie groups or the groups associated with so called Kac-Moody Lie algebras.

Lepowsky-Wilson [1] constructed an irreducible representation of the affine Lie algebra $A_1^{(1)}$, the central extension of $sl(2; C[t, t^{-1}])$, on the infinite-dimensional vector space $C[x_1, x_3, x_5, \cdots]$, which has the constant function as a highest weight vector. In their construction, the "vertex operator"

$$X(p)=\exp\left(2\sum x_j p^j\right)\exp\left(-2\sum\frac{1}{j}\frac{\partial}{\partial x_j}p^{-j}\right)\qquad (j\text{ odd}>0)$$

plays a crucial role. We noticed that this vertex operator is nothing but the infinitesimal Bäcklund transformation for the KdV equation in soliton theory. Thus, the group $A_1^{(1)}$ is the transformation group of the hierarchy of the KdV equations and the space of associated $\tau$ functions coincides with the orbit of the highest weight vector. This fact is also true for other soliton-type equations. In the case of the KP equation, the Lie algebra $\mathfrak{gl}(\infty)$ operates on $C[x_1, x_2, \cdots]$ through the vertex operator and the space of $\tau$ functions coincides with the orbit of the constant function. This corresponds to the remarkable discovery of M. Sato and Y. Sato [2] that the space of $\tau$ functions of the KP equation is the infinite-dimensional Grassmann manifold.

The hierarchy of the KdV equation is subholonomic in the sense that the general solution depends on finitely many arbitrary functions

*) Faculty of General Educations, Kyoto University.
**) Research Institute for Mathematical Sciences, Kyoto University.

of one variable. On the other hand, the KP hierarchy is sub-sub-holonomic in the sense that the general solution depends on arbitrary functions of two variables. This difference is reflected in the size of their transformation groups: finite groups, Lie groups, $sl(2; \boldsymbol{C}[t, t^{-1}])$ and $\mathfrak{gl}(\infty)$ in Galois theory, Picard-Vessiot theory, the KdV equations and the KP equations, respectively.

In search of other vertex operators which form a Lie algebra, we have found a new hierarchy of non-linear differential equations with $\mathrm{O}(\infty)$ as its transformation group. We call this the KP hierarchy of B-type (abbreviated by BKP hierarchy).

In § 1, we recall the construction of $A_1^{(1)}$ due to Lepowsky-Wilson.

We then show in § 2 that the space of $\tau$ functions for the KdV hierarchy is the orbit of the highest weight vector of this representation. In § 3 we discuss the reason why the vertex operator forms a Lie algebra. In §§ 4 and 5, the vertex operator for the KP hierarchy is introduced and we show that it forms the Lie algebra $\mathfrak{gl}(\infty)$. In § 6 similar considerations are applied to obtain the BKP hierarchy.

Detailed study of the BKP hierarchy will be given in the forthcoming papers [3]. The relation between soliton equations and Euclidean Lie algebras as their transformation groups will be discussed in [4].

We thank M. Sato and M. Jimbo for fruitful discussions and stimulation.

**§ 1.** We shall recall the construction of the representation of $A_1^{(1)}$, due to Lepowsky and Wilson. This is a representation on the infinite-dimensional vector space $V_0=\boldsymbol{C}[x_1, x_3, x_5, \cdots]$ described as follows. Consider the "vertex operator"

$$X(p)=\exp 2\tilde{\xi}(x, p) \exp -2\tilde{\xi}(\tilde{\partial}, p^{-1}) \tag{1}$$

where $\tilde{\xi}(x, p)=\sum_{j\ \mathrm{odd}>0} x_j p^j$, $\tilde{\xi}(\tilde{\partial}, p^{-1})=\sum_{j\ \mathrm{odd}>0}(1/j)(\partial/\partial x_j)p^{-j}$ and $\tilde{\partial}=(\partial/\partial x_1, (1/3)(\partial/\partial x_3)\cdots)$. If $\sum X_k p^k$ denotes the Laurent expansion of $X(p)$, then $\{X_k\}_{k\in \boldsymbol{Z}}$, together with $1$, $x_j$, $\partial/\partial x_j$ forms an infinite-dimensional Lie algebra isomorphic to $A_1^{(1)}$. The constant function is a highest weight vector of this representation, i.e. a vector annihilated by $X_k (k<0)$ and $\partial/\partial x_j$.

**§ 2.** In order to study the relation between Lepowsky-Wilson's representation of $A_1^{(1)}$ and the KdV hierarchy, let us consider the $\tau$ function of an $N$-soliton solution for the KdV equation in Hirota's bilinear formalism (Hirota [5])

$$\begin{aligned}\tau(x)&=\tau\left(x;\begin{matrix}a_1\cdots a_N\\ p_1\cdots p_N\end{matrix}\right)\\ &=1+\sum_{j=1}^{N} a_j \exp(2\tilde{\xi}(x, p_j))+\sum_{j<k} a_j a_k c_{jk} \exp(2\tilde{\xi}(x, p_j)\\ &\qquad+2\tilde{\xi}(x, p_k))+\cdots\end{aligned} \tag{2}$$

$$=\sum_{r=0}^{N}\sum_{1\leqq i_1<\cdots<i_r\leqq N} a_{i_1}\cdots a_{i_r}\Big(\prod_{1\leqq\nu<\mu\leqq r} c_{i_\nu i_\eta}\Big)\exp\Big(2\sum_{\nu=1}^{r}\tilde{\xi}(x,p_{i_\nu})\Big)$$

where $c_{i,j}=(p_i-p_j)^2/(p_i+p_j)^2$. Then, one can show by a direct calculation that

$$X(p)\exp\Big(2\sum_{i=1}^{r}\tilde{\xi}(x,p_i)\Big)$$
$$=\Big(\prod_{i=1}^{r}(p-p_i)^2\Big/\prod_{i=1}^{r}(p+p_i)^2\Big)\exp\Big(2\tilde{\xi}(x,p)+2\sum_{i=1}^{r}\tilde{\xi}(x,p_i)\Big).$$

This implies

$$X(p)^2\tau(x)=0$$

and hence

$$e^{aX(p)}\tau(x)=\tau(x)+aX(p)\tau(x)=\tau\Big(x\,;\begin{matrix}a & a_1\cdots a_N\\ p & p_1\cdots p_N\end{matrix}\Big). \tag{3}$$

Therefore $e^{aX(p)}$ transforms an $N$-soliton solution to an $(N+1)$-soliton solution. Since soliton solutions form a dense subset of the space of solutions for the KdV equation, $X(p)$ acts infinitesimally on the whole space. Thus we can state

**Proposition 1.** *The space of $\tau$ functions for the KdV equation is the orbit of the highest weight vector by the action of $A_1^{(1)}$.*

If $\tau(x)$ is a $\tau$ function then so is $\exp(c_0+\sum c_j x_j)\tau(x+a)$. This explains the fact that the Lie algebra contains 1, $x_j$ and $\partial/\partial x_j$.

§ 3. We shall calculate the commutation relation between $X(p)$ and $X(p')$. An easy calculation shows that

$$X(p)X(p')=((1-p'/p)^2/(1+p'/p)^2)Y(p,p')$$

where $Y(p,p')=\exp(2\tilde{\xi}(x,p)+2\tilde{\xi}(x,p'))\exp(-2\tilde{\xi}(\tilde{\partial},p^{-1})-2\tilde{\xi}(\tilde{\partial},p'^{-1}))$. Here $(1-p'/p)^2/(1+p'/p)^2$ is regarded as a power series in $p'/p$. Hence, we obtain

$$[X(p),X(p')]=f(p,p')Y(p,p')$$

with

$$f(p,p')=(1-p'/p)^2/(1+p'/p)^2-(1-p/p')^2/(1+p/p')^2.$$

Note that $f(p,p')$ does not vanish as a Laurent series in $p$ and $p'$ but vanishes as a rational function in $p$ and $p'$. Thus, if we introduce the $\delta$-function

$$\delta(t)=\sum_{n=-\infty}^{\infty}t^n=1/(1-t)+t^{-1}/(1-t^{-1}) \tag{4}$$

and its derivative $\delta'(t)=\sum nt^{n-1}=1/(1-t)^2-t^{-2}/(1-t^{-1})^2$, $f(p,p')$ is written as $4\delta'(-p'/p)-4\delta(-p'/p)$. By using $t\delta(t)=\delta(t)$, we obtain

$$[X(p),X(p')]$$
$$=4Y(p,-p)\delta'(-p'/p)+4\Big(p\frac{\partial Y(p,p')}{\partial p'}\Big|_{p'=-p}-Y(p,-p)\Big)\delta(-p'/p).$$

The relations $Y(p,-p)=1$ and

$$p\partial Y(p,p')/\partial p'|_{p'=-p}=\sum_{j\ \mathrm{odd}>0}\Big\{-jx_j(-p)^j+\frac{\partial}{\partial x_j}(-p)^{-j}\Big\}$$

give us the following identity.

$$[X_k, X_{k'}]=\begin{cases} -4(-1)^k(k+k')x_{k+k'} & \text{for } k+k' \text{ odd}>0 \\ 4(-1)^k(\partial/\partial x_{-k-k'}), & \text{for } k+k' \text{ odd}<0 \\ 4(-1)^{k+1}k & \text{for } k+k'=0 \\ 0 & \text{otherwise} \end{cases}.$$

§ 4. The preceding considerations on the KdV equation apply to the KP equation with minor changes. A soliton solution for the KP equation is given by

$$\begin{aligned}\tau(x)&=\tau\left(x\,;\begin{matrix} a_1 & \cdots a_N \\ p_1, q_1 & \cdots p_N, q_N\end{matrix}\right)\\ &=1+\sum a_j \exp \xi_j+\sum_{j<k} a_j a_k c_{jk} \exp(\xi_j+\xi_k)+\cdots\\ &=\sum_{r=0}^{N}\sum_{1\leqq i_1<\cdots<i_r\leqq N}\left(\prod_{\nu=1}^{r} a_{i_\nu}\right)\left(\prod_{\nu<\mu} c_{i_\nu i_\mu}\right)\exp\left(\sum_{\nu=1}^{r}\xi_{i_\nu}\right)\end{aligned}$$

where $\xi_i=\xi(x, p_i)-\xi(x, q_i)$, $\xi(x, p)=\sum_{j=1}^{\infty} x_j p^j$ and

$$c_{ij}=\frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}.$$

In this case, the $\tau$ function contains pairs of parameters $p_j$ and $q_j$. Bearing this in mind, we shall introduce the "vertex operator" with two variables $p$ and $q$

(6) $\quad X(p, q)=\exp(\xi(x, p)-\xi(x, q)) \exp(-\xi(\tilde{\partial}, p^{-1})+\xi(\tilde{\partial}, q^{-1}))$

where $\tilde{\partial}=(\partial/\partial x_1, (1/2)(\partial/\partial x_2), \cdots)$. Then we obtain

$$X(p, q)^2\tau(x)=0$$

and

$$e^{aX(p,q)}\tau(x)=\tau\left(x\,;\begin{matrix} a, & a_1 & \cdots a_N \\ p, q, & p_1, q_1 & \cdots p_N, q_N\end{matrix}\right).$$

Thus, as in the KdV case, we can state

**Proposition 2.** *$X(p, q)$ operates infinitesimally on the space of $\tau$ functions for the KP hierarchy.*

Note that the vertex operator $X(p)$ for the KdV hierarchy is obtained from $X(p, q)$ by putting $q=-p$. If we set $q=wp$ $(\omega^3=1)$, we obtain the vertex operator for the Boussinesq hierarchy. Further studies in this direction will be given in the forthcoming papers.

§ 5. Now we shall show that $X(p, q)$ generates the Lie algebra $\mathfrak{gl}(\infty)$. The line of calculation follows § 3. We have

(7) $$X(p, q)X(p', q')=\frac{(1-p'/p)(1-q'/q)}{(1-q'/p)(1-p'/q)}Y(p, q\,; p', q')$$

where

$$\begin{aligned}Y(p, q\,; p', q')=&\exp(\xi(x, p)+\xi(x, p')-\xi(x, q)-\xi(x, q'))\\ &\times\exp(-\xi(\tilde{\partial}, p^{-1})-\xi(\tilde{\partial}, p'^{-1})+\xi(\tilde{\partial}, q^{-1})+\xi(\tilde{\partial}, q'^{-1})).\end{aligned}$$

On the other hand, we have

$$\frac{(1-p'/p)(1-q'/q)}{(1-q'/p)(1-p'/q)}-\frac{(1-p/p')(1-q/q')}{(1-q/p')(1-p/q')}$$

$$=\frac{(1-q/p)(1-q'/p')}{1-q'/p}\delta(p'/q)-\frac{(1-q/p)(1-q'/p')}{1-q/p'}\delta(p/q').$$

Hence by setting $Z(p,q)=((q/p)/(1-q/p))X(p,q)$, we obtain

$$[Z(p,q),Z(p',q')]=Z(p,q')\delta(q/p')-Z(p',q)\delta(p/q').$$

If we develop $Z(p,q)$ as a Laurent series $\sum Z_{i,j}p^i q^{-j}$, then the $Z_{i,j}$'s satisfy

$$[Z_{i,j},Z_{i',j'}]=\delta_{j,i'}Z_{i,j'}-\delta_{j',i}Z_{i',j}.$$

This commutation relation is the same as that of $E_{i,j}$ (the matrix whose components are 0 except 1 at the $(i,j)$-component).

**Proposition 3.** (1) *$X(p,q)$ generates the Lie algebra $\mathfrak{gl}(\infty)$.* (2) *$\mathfrak{gl}(\infty)$ operates on the space of $\tau$ functions for the KP hierarchy.*

**§6.** As seen in the preceding section, the proof that $X(p,q)$ generates a Lie algebra relies on the fact that the commutator of $X(p,q)$ and $X(p',q')$ is a linear combination of $\delta$-functions. We found another example of vertex operators with the same property. In this case, the corresponding Lie algebra is $\mathfrak{o}(\infty)$, the infinite dimensional orthogonal Lie algebra.

Consider a vertex operator on $C[x_1,x_3,x_5,\cdots]$

$$X_B(p,q)=\exp(\tilde{\xi}(x,p)+\tilde{\xi}(x,q))\exp(-2\tilde{\xi}(\tilde{\partial},p^{-1})-2\tilde{\xi}(\tilde{\partial},q^{-1})) \tag{8}$$

where $\tilde{\xi}(x,p)$ and $\tilde{\xi}(\tilde{\partial},p^{-1})$ are as in (1). Then

$$Z_B(p,q)=\frac{1}{2}\frac{p-q}{p+q}X_B(p,q)\qquad\left(\frac{p-q}{p+q}=\frac{1}{2}\left(\frac{1-q/p}{1+q/p}+\frac{p/q-1}{p/q+1}\right)\right)$$

satisfies the relations

$$\begin{aligned}[Z_B(p,q),Z_B(p',q')]=&-Z_B(p,p')\delta(-q'/q)\\&+Z_B(p,q')\delta(-p'/q)+Z_B(q',q)\delta(-p/p')-Z_B(p',q)\delta(-p/q')\end{aligned} \tag{9}$$

and

$$Z_B(p,q)=-Z_B(q,p). \tag{10}$$

Hence if we develop $Z_B(p,q)$ as a Laurent series $\sum Z_{ij}p^i q^{-j}$, then $Z_{i,j}$ satisfies the same commutation relation as $(-1)^j E_{i,j}-(-1)^i E_{-j,-i}$, where $E_{i,j}$ is as in §5. Therefore, the $Z_{i,j}$'s generate the orthogonal Lie algebra with respect to the inner product $\xi^2=\sum_{i=-\infty}^{\infty}(-1)^i\xi_i\xi_{-i}$ on the space $C^\infty$ of $\xi=(\xi_i)_{i\in Z}$.

Successive application of $e^{aX_B(p,q)}$ to the constant function yields a class of $N$-solitons

$$\begin{aligned}\tau\left(x;\begin{matrix}a_1 & \cdots a_N\\ p_1,q_1\cdots p_N,q_N\end{matrix}\right)&=\exp\left(\sum a_jX_B(p_j,q_j)\right)1\\&=1+\sum_{j=1}^N a_j\exp\xi_j+\sum_{j<k}a_ja_kc_{jk}\exp(\xi_j+\xi_k)+\cdots\\&=\sum_{r=0}^N\sum_{i_1<\cdots<i_r}\left(\prod_{\nu=1}^r a_{i_\nu}\right)\left(\prod_{\nu<\mu}c_{i_\nu,i_\mu}\right)\exp\left(\sum_{\nu=1}^r\xi_{i_\nu}\right)\end{aligned} \tag{11}$$

where $\xi_i=\tilde{\xi}(x,p_i)+\tilde{\xi}(x,q_i)$ and

$$c_{jk}=\frac{(p_j-p_k)(p_j-q_k)(q_j-p_k)(q_j-q_k)}{(p_j+p_k)(p_j+q_k)(q_j+p_k)(q_j+q_k)}.$$

They satisfy a series of Hirota's bilinear differential equations, the first two of which read

$$(D_1^6-5D_1^3D_3-5D_3^2+9D_1D_5)\tau\cdot\tau=0,$$

$$(D_1^8+7D_1^5D_3-35D_1^2D_3^2-21D_1^3D_5-42D_3D_5+90D_1D_7)\tau\cdot\tau=0.$$

We shall call this hierarchy the BKP hierarchy. Further studies of this hierarchy will be given in the forthcoming papers [3]. Here we only remark that the restriction $q=-\omega p$ $(\omega^3=1)$ reduces the BKP hierarchy to the Sawada-Kotera hierarchy [6].

## References

[1] J. Lepowsky and R. L. Wilson: Commun. Math. Phys., **62**, 43 (1978).

[2] M. Sato: A lecture delivered at the symposium on Random Systems and Dynamical Systems held at RIMS, Kyoto Univ. on Jan. 7–9, 1981.

[3] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa: Transformation groups for soliton equations IV, V. RIMS preprint, nos. 359, 360 (1981).

[4] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa: Transformation groups for soliton equations. Euclidean Lie algebra and reduction of the *KP* hierarchy. ibid., no. 362 (1981).

[5] R. Hirota: Lect. Notes in Math., vol. 515, Springer (1976).

[6] K. Sawada and T. Kotera: Prog. Theo. Phys., **51**, 1355 (1974).