87. *Modular Forms of Degree n and Representation by Quadratic Forms. III*

Kloosterman's Method

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Kloosterman improved a result of Hecke about estimates of Fourier coefficients of cusp forms by using so-called Kloosterman sums. Our aim is to generalize his method to Siegel modular forms of degree 2 with two assumptions on exponential sums and to apply it to representations by quadratic forms.

Terminology and notations. Let H be the space of 2×2 complex symmetric matrices Z whose imaginary part is positive definite, and $\Gamma=Sp_2(\boldsymbol{Z})$ which acts on H discontinuously. Denote by $\mathfrak{F}$ the fundamental domain $\Gamma\backslash H$ by Siegel (p. 169 in [5]). By $\Gamma(\infty)$ we denote the subgroup $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\in\Gamma\right\}$ of Γ where 0 is the 2×2 zero matrix and $\mathfrak{S}$ stands for $\bigcup_{M\in\Gamma(\infty)} M\langle\mathfrak{F}\rangle$. By $\Lambda, \boldsymbol{Q}\Lambda$ and $\boldsymbol{R}\Lambda$ we denote the set of integral, rational and real symmetric 2×2 matrices respectively, and Λ^* stands for $\{(s_{ij})\in\boldsymbol{Q}\Lambda \mid s_{11}, s_{22}\in\boldsymbol{Z}, 2s_{12}\in\boldsymbol{Z}\}$. For $C, D\in M_2(\boldsymbol{Z})$, $(C, D)=1$ means that there exist matrices $A, B\in M_2(\boldsymbol{Z})$ such that $\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma$. σ stands for the trace of square matrices and $e(z)$ means $\exp(2\pi iz)$ for a complex number z.

We gather two assumptions and some lemmas.

Assumption 1. *Let c_1, c_2 be natural numbers with $c_1\mid c_2$ and $Y\in\boldsymbol{R}\Lambda$ positive definite. Then we assume*

$$\sum_{g_i}\Big|\sum_{s_i} e(s_1g_1/c_1+s_2g_2/c_1+s_4g_4/c_2)=O(c_1^2c_2^{1+\varepsilon}) \qquad \textit{for any } \varepsilon>0,$$

where g_1, g_2, s_1, s_2 run over $\boldsymbol{Z}/c_1\boldsymbol{Z}$ and g_4, s_4 run over $\boldsymbol{Z}/c_2\boldsymbol{Z}$ and moreover $\{s_i\}$ satisfies

$$\begin{pmatrix} s_1/c_1 & s_2/c_1 \\ s_2/c_1 & s_4/c_2\end{pmatrix}+\sqrt{-1}Y\in\mathfrak{S}.$$

Here O is independent of Y.

Assumption 2. *Let $C\in M_2(\boldsymbol{Z})$, $|C|\neq 0$. For $G_1, G_2\in\Lambda^*$ we put*

$$K(G_1, G_2; C)=\sum_D e(\sigma(AC^{-1}G_1+C^{-1}DG_2))$$

where D runs over $\{D\in M_2(\boldsymbol{Z}) \bmod C\Lambda \mid (C, D)=1\}$ and A is a matrix such that $\begin{pmatrix} A & * \\ C & D\end{pmatrix}\in\Gamma$. For these generalized Kloosterman sums we *assume for* $0<\kappa\leq 1/2$:

for natural numbers $c_1|c_2$ *and for* $G_1, G_2\in\Lambda^*$,

$$K\left(G_1, G_2; \begin{pmatrix} c_1 & 0 \\ 0 & c_2\end{pmatrix}\right)=O(c_1^2c_2^{1-\kappa+\varepsilon}(c_2, g)^\kappa) \qquad \textit{for any } \varepsilon>0,$$

where g is the (2, 2)*-entry of* G_2. ($\kappa=1/2$ is plausible.)

Let $C\in M_2(\boldsymbol{Z})$ with $|C|\neq 0$, and $\tau\in H$. For $S\in\boldsymbol{Q}\Lambda$ with $SC\in M_2(\boldsymbol{Z})$ we put

$$g(S; C, \tau)=\begin{cases}1 & \text{if } S+\tau\in\mathfrak{G},\\ 0 & \text{otherwise.}\end{cases}$$

Then we have $g(S; C, \tau)=\sum_G b(G; C, \tau)e(\sigma(SG))$ for some $b(G; C, \tau)\in\boldsymbol{C}$ where G runs over the representatives of $\Lambda^*/\mathfrak{S}(C)$, $\mathfrak{S}(C)=\{G\in\Lambda^* \mid \sigma(SG)\in\boldsymbol{Z}$ for $S\in\boldsymbol{Q}\Lambda$ which satisfies $SC\in M_2(\boldsymbol{Z})\}$.

Lemma 1. *If Assumption* 1 *is true, then for the above* C *and* τ *we have*

$$\sum_G |b(G; C, \tau)|=O(c_2^\varepsilon) \qquad \textit{for any } \varepsilon>0,$$

where G runs over $\Lambda^*/\mathfrak{S}(C)$ *and the elementary divisors of* C *are* $c_1, c_2>0$, $c_1|c_2$, *and O is independent of* τ.

For $G=(g_{ij})\in\Lambda^*$ we put $e(G)=(g_{11}, g_{22}, 2g_{12})$. Put

$$S=\left\{\begin{pmatrix} b\\ d\end{pmatrix} \middle| \, b, d\in\boldsymbol{Z}, (b, d)=1\right\}.$$

For a fixed natural number n we define an equivalence relation $\sim$ in S by the following:

$$\begin{pmatrix} b\\ d\end{pmatrix}\sim\begin{pmatrix} b'\\ d'\end{pmatrix} \text{ iff } \begin{pmatrix} b\\ d\end{pmatrix}\equiv w\begin{pmatrix} b'\\ d'\end{pmatrix} \bmod n \qquad \text{for an integer } w \text{ prime to } n.$$

Put $S(n)=S/\sim$; then we have

Lemma 2. *Let* $0<\kappa\leq 1/2$. *For* $G\in\Lambda^*$ *we have*

$$\sum_{x\in S(n)} (G[x], n)^\kappa=O(n^{1+\varepsilon}(e(G), n)^\kappa) \qquad \textit{for any } \varepsilon>0.$$

Let $T=\begin{pmatrix} t_1 & t_2\\ t_2 & t_4\end{pmatrix}\in\Lambda^*$ such that $0<t_1$, $2|t_2|\leq t_1$, $t_4\geq t_1$, and assume that t_1 is sufficiently large. We shall fix such a T once and for all in the following.

Lemma 3. *Let*

$$M=\begin{pmatrix} * & * \\ \begin{pmatrix} c & 0\\ 0 & 0\end{pmatrix}{}^tU & \begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix}U^{-1}\end{pmatrix}\in\Gamma$$

with $c\neq 0$, $U\in GL(2, \boldsymbol{Z})$. *If* $M\langle X+iT^{-1}\rangle\in\mathfrak{G}$ *for some* $X\in\boldsymbol{R}\Lambda$, *then the first column of* U *is equal to* $\pm\begin{pmatrix} 0\\ 1\end{pmatrix}$ *or* $\pm\begin{pmatrix} 1\\ n\end{pmatrix}$, $n\in\boldsymbol{Z}$.

Now let q, k be natural numbers ($k\geq 3$), and

$$f(Z)=\sum_{0\leq P\in\Lambda^*} a(P)e(\sigma(PZ))$$

be a modular form of degree 2, level q and weight k whose constant term vanishes at every cusp, that is,

(i) $f(Z)$ is holomorphic on H,

(ii) putting $(f|M)(Z)=|CZ+D|^{-k}f(M\langle Z\rangle)$ for

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma,\qquad (f|M)(Z)=\sum_{0\leq P\in\Lambda^*}a_M(P)e(\sigma(PZ)/q)$$

with $a_M(0)=0$ for $M\in\Gamma$,

(iii) if $M\in\Gamma$ satisfies $M\equiv 1_2 \bmod q$, then $f|M=f$ follows.

Put $E=\{(x_{ij})\in \boldsymbol{R}\Lambda\,|\,0\leq x_{ij}<q\}$ and $E(M)=\{X\in E\,|\,M\langle X+iT^{-1}\rangle\in\mathfrak{G}\}$ for $M\in\Gamma(\infty)\backslash\Gamma$; then the measure of $E(M)\cap E(N)$ becomes 0 if $\Gamma(\infty)M\neq\Gamma(\infty)N$. Hence we have

$$a(T)=q^{-3}\exp(4\pi)\sum_{\substack{M\in\Gamma(\infty)\backslash\Gamma\\ M\notin\Gamma(\infty)}}\alpha(M),$$

where

$$\alpha(M)=\alpha(C,D)=\int_{x\in E(M)}f(X+iT^{-1})e(-\sigma(TX))dX,\qquad M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}.$$

Proposition 1. *Let $C,\tilde{D}\in M_2(\boldsymbol{Z})$ and assume that $|C|\neq 0$ and there is an element $\begin{pmatrix}\alpha & \beta\\ C & \delta\end{pmatrix}\in\Gamma$ such that $\delta\equiv\tilde{D}\bmod q$. We put $\tau=\tau(\theta)$ $=-{}^tC^{-1}(\theta+iT^{-1})^{-1}C^{-1}$ for $\theta\in\boldsymbol{R}\Lambda$ and*

$$\left(f\left|\begin{pmatrix}\alpha & \beta\\ C & \delta\end{pmatrix}^{-1}\right.\right)(Z)=\sum_{0\leq P\in\Lambda^*}a'(P)e(\sigma(PZ)/q).$$

Then we have

$$\begin{aligned}\sum_{\substack{D\\ \{D\equiv\tilde{D}\bmod q\\ (C,D)=1}}\alpha(C,D)
&=\sum_{D_i}[\Lambda\cap qC^{-1}M_2(\boldsymbol{Z}):q\Lambda]\,|C|^{-k}\int_{A_iC^{-1}+\tau\in\mathfrak{G}}|\theta+iT^{-1}|^{-k}\sum_{\substack{P\\(*)}}a'(P)\\
&\qquad\times e(\sigma(P\tau)/q)e(-\sigma(T\theta))e(\sigma(PA_iC^{-1}q^{-1}+TC^{-1}D_i))d\theta\\
&=[\Lambda\cap qC^{-1}M_2(\boldsymbol{Z}):q\Lambda]\,|C|^{-k}\int_\theta|\theta+iT^{-1}|^{-k}\sum_{\substack{P\\(*)}}a'(P)e(\sigma(P\tau)/q)\\
&\qquad\times e(-\sigma(T\theta))\sum_G b(G;C,\tau)S\left(G,P,T;C,\begin{pmatrix}\alpha & \beta\\ C & \delta\end{pmatrix}\right)d\theta,\end{aligned}$$

where

$$S\left(G,P,T;C,\begin{pmatrix}\alpha & \beta\\ C & \delta\end{pmatrix}\right)=\sum_{D_i}e(\sigma(A_iC^{-1}G+PA_iC^{-1}q^{-1}+TC^{-1}D_i)).$$

Here D_i runs over the set $\{D\in M_2(\boldsymbol{Z})\bmod\{CS\,|\,S\in\Lambda, CS\equiv 0\bmod q\}\,|\,(C,D)=1,\ D\equiv\tilde{D}\bmod q\}$ and A_i is a matrix which satisfies

$$M=\begin{pmatrix}A_i & *\\ C & D_i\end{pmatrix}\equiv\begin{pmatrix}\alpha & \beta\\ C & \delta\end{pmatrix}\bmod q,\ M\in\Gamma.$$

G runs over $\Lambda^*/\mathfrak{S}(C)$ as in Lemma 1. $P\in\Lambda^*$ satisfies the condition $(*)$:

$(*)$ *$\sigma(PS)\equiv 0\bmod q$ for $S\in\Lambda$ such that ${}^tCS\equiv 0\bmod q$.*

The above $S\left(G,P,T;C,\begin{pmatrix}\alpha & \beta\\ C & \delta\end{pmatrix}\right)$ can be represented by the above generalized Kloosterman sums.

Proposition 2. *Let*

$$C=U\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}V,\ c_1\,|\,c_2,\ U,\ V\in GL(2,Z)$$

for natural numbers c_1, c_2, *and* $\tilde{D}\in M_2(Z)$. *Under Assumptions* 1, 2 *we have*

$$\Big|\sum_{\substack{D \\ \{D\equiv\tilde{D} \bmod q \\ (C,D)=1}} \alpha(C,D)\Big|$$

$$\ll |C|^{-k}c_1^2c_2^{1-\kappa+\varepsilon}(c_2,\ T[V^{-1}]_4)^{\kappa}\int_{\theta}\|\theta+iT^{-1}\|^{-k}\exp(-\kappa_1 m(Im\tau))d\theta$$

$$\ll |T|^{k-3/2}c_1^{2-k}c_2^{1-\kappa-k+\varepsilon}(c_2,\ T[V^{-1}]_4)^{\kappa}$$

$$\times\begin{cases} c_2^{k-2}t_1^{1-k/2} & \text{if } c_2\le\sqrt{t_1}, \\ 1 & \text{if } c_2>\sqrt{t_1}, \end{cases}$$

where ε *is any positive number,* $T[V^{-1}]_4$ *stands for the* (2, 2)*-entry of* $T[V^{-1}]$, κ_1 *is a positive constant and* $\tau=-{}^tC^{-1}(\theta+iT^{-1})^{-1}{}^tC^{-1}$ *and*

$$m(Y)=\min_{\substack{x\in Z^2 \\ x\neq 0}} Y[x] \qquad \text{for positive } Y\in RA.$$

By using Lemma 2, we have

Proposition 3. *Let* $\tilde{D}\in M_2(Z)$. *Under Assumptions* 1, 2 *we have*

$$\sum_{\substack{M=\binom{A\ B}{C\ D}\in\Gamma(\infty)\backslash\Gamma \\ |C|\neq 0,\, D\equiv\tilde{D} \bmod q}} \alpha(C,D)=O(t_1^{(3-k)/2-\kappa/2+\varepsilon}|T|^{k-3/2}).$$

By using Lemma 3 and a method in [3] we have

Proposition 4. *Let* $\tilde{d}\in Z$. *Then we have*

$$\sum\alpha\left(\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}{}^tU,\ \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}U^{-1}\right)=O\left(t_1^{2-k+\varepsilon}|T|^{k-3/2}\left(\sum_{r|t_4} r^{2-k}d(r)\log r\right)\right),$$

where c, d *run over* Z *so that* $(c,d)=1$, $d\equiv\tilde{d} \bmod q$ *and* $c\neq 0$, *and*

$$U\in GL(2,Z)/\left\{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\,\middle|\, n\in Z\right\},\quad \text{and}\quad d(r)=\sum_{n|r}1.$$

Summing up, we have

Theorem. *Let* $f(Z)=\sum_{0\le P\in A^*}a(P)e(\sigma(PZ))$ *be a modular form of degree* 2, *level* q *and weight* $k\ge 3$ *whose constant term vanishes at each cusp. Under Assumptions* 1, 2 *we have*

$$a(T)=O(t_1^{(3-k)/2-\kappa/2+\varepsilon}|T|^{k-3/2})$$

$$\times\begin{Bmatrix} 1 & k\ge 4 \\ \sum_{r|t_4} r^{-1}d(r)\log r & k=3 \end{Bmatrix},\qquad \text{if } t_1\gg 0,$$

where $T=\begin{pmatrix} t_1 & t_2 \\ t_2 & t_4 \end{pmatrix}\in A^*$ *with* $0<t_1$, $2|t_2|\le t_1$, $t_1\le t_4$.

Corollary. *Let* A, B *be integral symmetric positive definite matrices of degree* 6, 2 *respectively. Suppose that Assumptions* 1, 2 *are true and that* $A[X]=B$ *is soluble in* $M_{6,2}(Z_p)$ *for every prime* p, *and that for any fixed number* t, $p^t\nmid|B|$ *if a prime* p *divides* $2|A|$. *Then* $A[X]=B$ *is soluble in* $M_{6,2}(Z)$ *if* b_1 *is sufficiently large and either* $|B|$

$<\exp(b_1^{\varepsilon/5})$ *or* $\sum_{r|b_4} r^{-1}d(r)\log r<t$ *where* $B=\begin{pmatrix} b_1 & b_2 \\ b_2 & b_4 \end{pmatrix}$ *and* $0<b_1$, $2|b_2| \leq b_1$, $b_1 \leq b_4$.

Remark. $\sum_{r|b} r^{-1}d(r)\log r \ll \min(d(b), (\log b)^2)$. If degree of $A \geq 7$, then it is known that the local solubility of $A[X]=B$ yields the global solubility if $m(B)\gg 0$.

Detailed proofs will appear elsewhere.

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