82. "Borel" Lines for Meromorphic Solutions of the Difference Equation $y(x+1)=y(x)+1+\lambda/y(x)$

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1. Introduction. In connection with the iteration of analytic functions, Kimura [1], [2] considered the equation

(E) $$y(x+1)=y(x)+1+\lambda/y(x), \qquad \lambda\neq 0,$$

and obtained a meromorphic solution $\phi(x)$ such that

(1.1) $$\begin{cases} \phi(x)\sim x\left[1+\sum_{j+k\geqq 1} p_{jk}x^{-j}\left(\dfrac{\log x}{x}\right)^k\right] \qquad (p_{01}=\lambda) \\ \text{in the domain } D_l(R,\varepsilon)=\left\{|x|>R,\ |\arg x-\pi|<\dfrac{\pi}{2}-\varepsilon\right\}\cup\{\operatorname{Im}[xe^{-i\varepsilon}] \\ >R\}\cup\{\operatorname{Im}[xe^{i\varepsilon}]<-R\}, \text{ where } p_{10}=c \text{ is an arbitrarily prescribed constant, } \varepsilon>0, \text{ and } R \text{ is a sufficiently large number} \\ \text{depending on } c \text{ and } \varepsilon. \end{cases}$$

We studied some properties of the solution ϕ(x) in [3] and, especially, proved that there is a horizontal line $L=\{\operatorname{Im} x=\eta\}$ such that, for any $\delta>0$, in the half strip

(1.2) $$\{x\,;\,|\operatorname{Im} x-\eta|<\delta,\ \operatorname{Re} x>0\},$$

$\phi(x)$ takes every value infinitely often if $\lambda\neq 1$, and $\phi(x)$ takes every value other than -1 if $\lambda=1$.

We will call such a line as a *"Borel" line* for $\phi(x)$ [4]. It would be natural to inquire how many "Borel" lines may appear for $\phi(x)$.

Our aim in this note is to answer (partially) to this question. We will prove the following

Theorem. *Suppose λ is real in the equation* (E).

(i) *If $\lambda\leqq 1/4$, then there is only one "Borel" line for $\phi(x)$.*

(ii) *If $\lambda>1/4$, then there are at least two "Borel" lines for $\phi(x)$.*

2. Proof of Theorem (i). Let x_0 be a zero point of $\phi(x)$: $\phi(x_0)=0$. Write $x_n=x_0-n$, $n=0,1,\cdots$. Then $\phi(x_1)$ satisfies $0=\phi(x_1)+1+\lambda/\phi(x_1)$, i.e.,

(2.1) $$\phi(x_1)=\frac{1}{2}[-1\pm\sqrt{1-4\lambda}].$$

More generally

(2.2) $$\phi(x_n)=\frac{1}{2}[-(1-\phi(x_{n-1})\pm\sqrt{(1-\phi(x_{n-1}))^2-4\lambda}], \qquad n=1,2,\cdots.$$

We consider the following two cases:

(a) When $0<\lambda\leqq 1/4$; (b) When $\lambda<0$.

(a) When $0<\lambda\leqq 1/4$.

In this case, $\phi(x_1)\leqq 0$ from (2.1). Suppose $\phi(x_{n-1})$ be real and $\leqq 0$. Then

$$(1-\phi(x_{n-1}))^2-4\lambda\geqq 0,$$

hence from (2.2) we know that $\phi(x_n)$ is real and $\leqq 0$. Thus, $\phi(x_n)$, $n=1, 2, \cdots$ are all real and $\leqq 0$ in this case.

(b) When $\lambda<0$.

Obviously, $\phi(x_n)$, $n=1, 2, \cdots$ are all real.

Thus, *in both cases* (a) *and* (b), $\phi(x_n)$ *are all real for* $n=1, 2, \cdots$.

If n is sufficiently large, then by (1.1) we have

$$\phi(x_n)\sim x_n+c+\lambda\log x_n \qquad (c=p_{10}). \tag{2.3}$$

Since $\phi(x_n)$ are all real, we have

$$\operatorname{Im}[x_n+c+\lambda\log x_n]=\operatorname{Im} x_0+\operatorname{Im} c+\lambda\arg(x_0-n)\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{2.4}$$

Since $\arg(x_0-n)\to\pi$ as $n\to\infty$, we know by (2.4) that zero points of $\phi(x)$ must lie on a horizontal line

$$L=\{x\,;\ \operatorname{Im} x=-\operatorname{Im} c-\lambda\pi\}.$$

Therefore, any other line than L can not be a "Borel" line, because for sufficiently small $\delta>0$, the half-strip (1.2) can not contain any zero points of $\phi(x)$.

3. Proof of Theorem (ii). Let x_0 and x_n be the same as in § 2. Put $\phi(x_n)=u_n+iv_n$ and write

$$A_n=(u_n-1)^2-v_n^2-4\lambda, \qquad B_n=2(u_n-1)v_n. \tag{3.1}$$

Then by (2.2) we obtain

$$u_{n+1}=\frac{1}{2}\left[(u_n-1)\pm\sqrt{\frac{1}{2}\{\sqrt{A_n^2+B_n^2}+A_n\}}\right], \tag{3.2}$$

$$v_{n+1}=\frac{1}{2}\left[v_n\pm\gamma_n\sqrt{\frac{1}{2}\{\sqrt{A_n^2+B_n^2}-A_n\}}\right], \tag{3.3}$$

where γ_n is the sign of B_n.

Since $\lambda>1/4$, $\phi(x_1)$ is not real. Suppose $\phi(x_n)$ is not real. Then $\phi(x_{n+1})$ is a root of the quadratic equation

$$t^2+(1-\phi(x_n))t+\lambda=0. \tag{3.4}$$

Since λ is real, none of the roots of (3.4) are real. Thus, none of $\phi(x_n)$, $n=1, 2, \cdots$, are real.

If n is sufficiently large, then by (2.3) $u_n-1\sim\operatorname{Re}[x_0-n]<0$, hence we take the minus sign before $\sqrt{}$-symbol in (3.2) and (3.3), i.e.,

$$u_{n+1}=\frac{1}{2}\left[(u_n-1)-\sqrt{\frac{1}{2}\{\sqrt{A_n^2+B_n^2}+A_n\}}\right], \tag{3.2'}$$

$$v_{n+1}=\frac{1}{2}\left[v_n-\gamma_n\sqrt{\frac{1}{2}\{\sqrt{A_n^2+B_n^2}-A_n\}}\right]. \tag{3.3'}$$

By (3.3′) we have, supposing that $|u_n-1|$ is sufficiently large,

$$v_{n+1}=v_n[1+\lambda/(u_n-1)^2+\cdots], \tag{3.5}$$

and $v_{n+1}/v_n>1$ since $\lambda>0$, hence $|v_n|$ increases with n if $|u_n|$ is sufficiently large. Thus $v_n\to v_\infty\neq 0$ as $n\to\infty$. $v_\infty\neq\infty$ since [3, p. 102]

$$|\phi(x)/x-1|<1/2 \quad \text{for } |\text{Im } x|>R' \quad (\geqq R \text{ in (1.1)}) \tag{3.6}$$

and hence $|\text{Im } x_0|\leqq R'$ for any zero point x_0 of $\phi(x)$. Therefore

$$\text{Im }[x_n+c+\lambda\log x_n]=\text{Im } x_0+\text{Im } c+\lambda\arg(x_0-n)\longrightarrow v_\infty\neq 0,\ \infty,$$

whence we know that, if we write

$$\eta_0=-\text{Im } c-\lambda\pi+v_\infty,$$

then the zero point x_0 lies on the line

$$L(\eta_0)=\{x\,;\ \text{Im } x=\eta_0\}.$$

Thus the pole (x_0+1) of $\phi(x)$ also lies on $L(\eta_0)$. Take $\delta>0$ arbitrarily.

For any complex number b, let $x_0(b)$ be a b-point of $\phi(x)$: $\phi(x_0(b))=b$, and $x_n(b)=x_0(b)-n$. If n is sufficiently large, then by (2.3) $\phi(x_n(b))\sim x_n(b)+c+\lambda\log x_n(b)$, which is large. Thus the value $\phi(x_n(b))$ is taken at a point $x'(b)$ in the neighborhood $\{x\,;\ |x-(x_0+1)|<\delta\}$ of the pole (x_0+1). Thus, in the strip

$$H(\eta_0\,;\ \delta)=\{x\,;\ |\text{Im } x-\eta_0|<\delta\}$$

contains a b-point $x_0'(b)=x'(b)+n'$ for some positive integer n'. Therefore, the strip $H(\eta_0\,;\ \delta)$ contains infinitely many b-points of $\phi(x)$. Since b is any complex number, we know that $L(\eta_0)$ is a "Borel" line for $\phi(x)$.

Since λ is real, we must have another "Borel" line

$$\{x\,;\ \text{Im } x=-\text{Im } c-\lambda\pi-v_\infty\},$$

and our theorem is proved. (We note that $v_\infty\neq 0$.)

Remark. It is easy to see that

$$\sqrt{\frac{1}{2}\{\sqrt{A_n^2+B_n^2}+A_n\}}<|u_n-1|,$$

$$\sqrt{\frac{1}{2}\{\sqrt{A_n^2+B_n^2}-A_n\}}>|v_n|.$$

Suppose $u_n-1<0$ and $|u_n-1|$ is very large. If we take the plus-sign in front of $\sqrt{\ }$-symbol in (3.2) and (3.3), then

$$u_{n+1}<0,\ v_{n+1}v_n<0, \quad \text{and } |u_{n+1}|,\ |v_{n+1}| \quad \text{are very small.}$$

If we start from these (u_{n+1}, v_{n+1}), then we will obtain very small $|v_\infty|$. From this consideration, it is quite plausible that there might be infinitely many "Borel" lines $L(\eta_n)$ and $\eta_n\to\eta^*=-\text{Im } c-\lambda\pi$.

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