79. A Remark on the Hadamard Variational Formula. II

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§ 1. **Introduction.** Let $f(x)$ be a real-valued C^∞-function of x in R^n. Let $\Omega_t=\{x\in\mathrm{R}^n \mid f(x)<t\}$ for any real t. Then its boundary is $\gamma_t=\{x\in\mathrm{R}^n \mid f(x)=t\}$. We assume the following assumptions for f:

(A.1) Ω_2 is a bounded domain diffeomorphic to the unit disc.

(A.2) All values $t\in[-2,0)\cup(0,2]$ are regular values of f.

(A.3) Ω_2 contains only one critical point x^0 of f, where $f(x^0)=0$ and f has the non-degenerate Hessian of the index $n-1$.

For any $t\in[-1,0)\cup(0,1]$, we consider the following boundary value problem for u:

$$(\lambda-\Delta)u(x)=w(x), \qquad \text{for } x\in\Omega_t, \tag{1.1}$$

$$\frac{\partial}{\partial\nu}u(x)=0, \qquad \text{for } x\in\gamma_t, \tag{1.2}$$

where ν is the outer unit normal to γ_t and $\lambda\in\mathrm{C}$. If $\lambda>0$, u is uniquely determined by w and we put $u(x)=N_t(\lambda)w(x)$. Let $N_t(\lambda,x,y)$ be the integral kernel function of the mapping: $w\mapsto N_t(\lambda)w$, i.e.,

$$N_t(\lambda)w(x)=\int_{\Omega_t} N_t(\lambda,x,y)w(y)dy. \tag{1.3}$$

It is well known from the Hadamard variational formula that the function $N_t(\lambda,x,y)$ is continuously differentiable with respect to t if $t\neq 0$ and $x,y\in\Omega_{-1}$. The Hadamard variational formula implies that

$$\begin{aligned}\frac{d}{dt}N_t(\lambda,x,y)&\\ =\int_{\gamma_t} N_t(\lambda,z,y)&N_t(\lambda,z,x)\frac{1}{|\mathrm{grad}\, f(z)|}d\sigma(z)\\ &+\int_{\gamma_t}\langle \nabla'_z N_t(\lambda,z,y),\ \nabla'_z N_t(\lambda,z,x)\rangle\frac{1}{|\mathrm{grad}\, f(z)|}d\sigma(z)\end{aligned} \tag{1.4}$$

where $d\sigma$ is the volume element of γ_t, $\nabla'_z N_t(\lambda,z,y)$ denotes the component tangent to γ_t of the gradient vector of $N_t(\lambda,z,y)$ with respect to z and $\langle\ ,\ \rangle$ denotes the inner product in the tangent vector space to γ_t. See, for instance, Hadamard [6], Aomoto [1], Peetre [8] and Fujiwara-Ozawa [3].

For any small $\varepsilon>0$, we have

$$N_1(\lambda,x,y)-N_\varepsilon(\lambda,x,y)=\int_\varepsilon^1\frac{d}{d\tau}N_\tau(\lambda,x,y)d\tau \tag{1.5}$$

if x and $y \in \Omega_{-1}$. Hence the following natural question arises:

(Q) Can one replace ε in (1.5) by -1?

This is not a trivial question, because Ω_t is connected for $t>0$ but Ω_t has two connected components for $t<0$. Cf. Milnor [7].

The aim of this note is to give an affirmative answer to the question (Q) above:

Theorem. *If $\lambda>0$ and $x, y \in \Omega_{-1}$, we have*

$$\int_{-1}^{1} \left| \frac{d}{dt} N_t(\lambda, x, y) \right| dt < \infty \tag{1.6}$$

and

$$N_1(\lambda, x, y) - N_{-1}(\lambda, x, y) = \int_{-1}^{1} \frac{d}{dt} N_t(\lambda, x, y) dt. \tag{1.7}$$

Remark. A similar formula for the Green kernels of Dirichlet problem was discussed earlier in [2].

§ 2. Weak solution to the boundary value problems. Let

$$H^m(\Omega_t) = \{w \in L^2(\Omega_t) \mid D^\alpha w \in L^2(\Omega_t) \text{ for } |\alpha| \leq m\}$$

be the Sobolev space of order $m \geq 0$. Let $w \in L^2(\Omega_t)$. Then the solution $u(x)$ of the boundary value problem (1.1), (1.2) is characterized as follows: $u \in H^1(\Omega_t)$ and for any $\varphi \in H^1(\Omega_t)$,

$$\int_{\Omega_t} [\nabla u(x) \nabla \varphi(x) + \lambda u(x) \varphi(x)] dx = \int_{\Omega_t} w(x) \varphi(x) dx. \tag{2.1}$$

This formulation is valid even in the case $t=0$. We can thus define $N_t(\lambda, x, y)$ for $t=0$ too. We have, from (2.1), well known a priori estimate for $u = N_t(\lambda) w$.

Lemma 1. *For any $t \in [-1, 1]$ and $w \in L^2(\Omega_t)$, we have*

$$\int_{\Omega_t} |\nabla N_t(\lambda) w(x)|^2 \, dx + \lambda \int_{\Omega_t} |N_t(\lambda) w(x)|^2 \, dx \leq \lambda^{-1} \int_{\Omega_t} |w(x)|^2 \, dx. \tag{2.2}$$

§ 3. Proof of the theorem. If $t<0$, Ω_t has two connected components, which we denote by Ω_t^1 and Ω_t^2. We may assume that $\Omega_t^1 \subset \Omega_0^1$ and $\Omega_t^2 \subset \Omega_0^2$ for $t<0$. Thus, the space $H^1(\Omega_t)$ is the direct sum

$$H^1(\Omega_t) = H^1(\Omega_t^1) \oplus H^1(\Omega_t^2).$$

Since each of Ω_t^1 and Ω_t^2 has strong cone property, there exists a linear continuous extension map $H^1(\Omega_t^1) \to H^1(\mathrm{R}^n)$. Composing this with the restriction map $H^1(\mathrm{R}^n) \to H^1(\Omega_0^1)$, we have a linear continuous extension map $H^1(\Omega_t^1) \to H^1(\Omega_0^1)$. Similarly we have a continuous linear extension map $H^1(\Omega_t^2) \to H^1(\Omega_0^2)$. Thus we have

Lemma 2. *If $t<0$, there exists a linear extension map $E_t : H^1(\Omega_t) \to H^1(\Omega_0)$ such that for any $u \in H^1(\Omega_t)$*

$$\|E_t u\|_{H^1(\Omega_0)} \leq K \|u\|_{H^1(\Omega_t)}. \tag{3.1}$$

Here K is a positive constant independent of t and u.

Lemma 3. *For any $w \in L^2(\Omega_{-1})$, we have*

$$\lim_{t \uparrow 0} E_t N_t(\lambda) w = N_0(\lambda) w$$

in the strong topology of $L^2(\Omega_0)$ and in the weak topology of $H^1(\Omega_0)$.

Proof. By Lemmas 1 and 2, $\{E_t N_t(\lambda)w\}_{t<0}$ forms a bounded set of $H^1(\Omega_0)$. Let $\{t_j\}_{j=1}$ be any sequence such that $t_j \nearrow 0$. Then, there exists a subsequence $\{s_j\}$, such that $E_{s_j}N_{s_j}(\lambda)w=u_j$ converges to a certain function $g \in H^1(\Omega_0)$ strongly in $L^2(\Omega_0)$ and weakly in $H^1(\Omega_0)$. We have only to prove that $g=N_0(\lambda)w$, which is independent of the sequence $\{t_j\}$. Let φ be an arbitrary function in $H^1(\Omega_0)$. Then its restriction to Ω_t, $t<0$, belongs to $H^1(\Omega_t)$. Thus, if $t<0$, we have, from (2.1),

$$\int_{\Omega_{s_j}} [\nabla u_j(x)\nabla\varphi(x)+\lambda u_j(x)\varphi(x)]\,dx=\int_{\Omega_{s_j}} w(x)\varphi(x)dx. \tag{3.2}$$

The Schwartz' inequality gives the estimate

$$\left|\int_{\Omega_0\setminus\Omega_{s_j}} [\nabla u_j(x)\nabla\varphi(x)+\lambda u_j(x)\varphi(x)]\,dx\right| \le \left[\int_{\Omega_0\setminus\Omega_{s_j}} \{|\nabla u_j(x)|^2+\lambda\,|u_j(x)|^2\}dx\right]^{1/2} \times\left[\int_{\Omega_0\setminus\Omega_{s_j}} \{|\nabla\varphi(x)|^2+\lambda\,|\varphi(x)|^2\}dx\right]^{1/2}. \tag{3.3}$$

The right hand side tends to 0 as j goes to ∞. Hence

$$\begin{aligned}\int_{\Omega_0} w(x)\varphi(x)dx&=\lim_{j\to\infty}\int_{\Omega_{s_j}} w(x)\varphi(x)dx\\&=\lim_{j\to\infty}\int_{\Omega_{s_j}}[\nabla u_j(x)\nabla\varphi(x)+\lambda u_j(x)\varphi(x)]dx\\&=\lim_{j\to\infty}\int_{\Omega_0}[\nabla u_j(x)\nabla\varphi(x)+\lambda u_j(x)\varphi(x)]dx\\&=\int_{\Omega_0}[\nabla g(x)\nabla\varphi(x)+\lambda g(x)\varphi(x)]dx.\end{aligned}$$

Thus we have $g=N_0(\lambda)w$. This proves Lemma 3.

For any $w \in L^2(\Omega_0)$, $N_t(\lambda)w \in H^1(\Omega_t)$ for $t>0$. Let $R_0N_t(\lambda)w$ be its restriction to Ω_0. Then $R_0N_t(\lambda)w \in H^1(\Omega_0)$.

Lemma 4. *For any $w \in L^2(\Omega_0)$, we have*

$$\lim_{t\searrow 0} R_0N_t(\lambda)w=N_0(\lambda)w$$

in the strong topology of $L^2(\Omega_0)$ and in the weak topology of $H^1(\Omega_0)$.

Proof. First note that $\{R_0N_t(\lambda)w\}_{t>0}$ forms a bounded set of $H^1(\Omega_0)$. Let $\{t_j\}_{j=1}^{\infty}$ be any sequence such that $t_j\searrow 0$. Then, there exists a subsequence $\{s_j\}_j$ such that $R_0N_{s_j}(\lambda)w=v_j$ converges to a certain function $g \in H^1(\Omega_0)$ weakly in $H^1(\Omega_0)$ and strongly in $L^2(\Omega_0)$. We have only to prove that $g=N_0(\lambda)w$, which is independent of the sequence $\{s_j\}_j$. Let $\varphi \in H^1(R^n)$. Then as in the proof of Lemma 3, we have

$$\begin{aligned}\int_{\Omega_0} w(x)\varphi(x)dx&=\lim_{j\to\infty}\int_{\Omega_{s_j}}[\nabla v_j(x)\nabla\varphi(x)+\lambda v_j(x)\varphi(x)]dx\\&=\int_{\Omega_0}[\nabla g(x)\nabla\varphi(x)+\lambda g(x)\varphi(x)]dx.\end{aligned}$$

In the case $n\geq 3$, the restriction mapping $H^1(R^n)\to H^1(\Omega_0)$ is surjective. In the case $n=2$, it is not surjective but its image is dense in $H^1(\Omega_0)$. Cf. Grisvard [5]. Therefore for any $\varphi \in H^1(\Omega_0)$, we have

$$\int_{\Omega_0} w(x)\varphi(x)\,dx=\int_{\Omega_0}[\nabla g(x)\nabla\varphi(x)+\lambda g(x)\varphi(x)]dx.$$

This means that $g=N_0(\lambda)w$. Lemma 4 is proved.

We can prove convergence of the kernel function $N_t(\lambda, x, y)$ itself as $t\to 0$.

Lemma 5. *Assume that x and $y\in\Omega_0$. Then,*

(i) $\lim_{t\uparrow 0} N_t(\lambda, x, y)-N_0(\lambda, x, y)=0$,

(ii) $\lim_{t\downarrow 0} N_t(\lambda, x, y)-N_0(\lambda, x, y)=0$.

Proof. Let $\Gamma(z)$ be a parametrix of $(\lambda-\Delta)$, i.e.,

$$(\lambda-\Delta)\Gamma(z)=\delta(z)+\omega(z),$$

where $\omega(z)\in C_0^\infty(R^n)$. We may assume that $\Gamma(z-x)$ and $\Gamma(z-y)$ vanish if $z\notin\Omega_0$. Let $H_t(\lambda, x, y)=N_0(\lambda, x, y)-N_t(\lambda, x, y)$. Then

$$(\lambda-\Delta)H_t(\lambda, x, y)=0. \tag{3.4}$$

Therefore,

$$\begin{aligned}H_t(\lambda, x, y)&=\int_{\Omega_0}\int_{\Omega_0} H_t(\lambda,\xi,\eta)[(\lambda-\Delta_\xi)\Gamma(\xi-x)-\omega(\xi-x)]\\&\qquad\times[(\lambda-\Delta_\eta)\Gamma(\eta-y)-\omega(\eta-y)]d\xi d\eta\\&=\int_{\Omega_0}\int_{\Omega_0} H_t(\lambda,\xi,\eta)\omega(\xi-x)\omega(\eta-y)d\xi d\eta.\end{aligned}$$

The last equality results from (3.4) and integration by parts. Since $\omega(\xi-x)$ and $\omega(\eta-y)$ are functions in $L^2(\Omega_0)$, Lemma 3 proves (i). Similarly (ii) follows from Lemma 4.

Lemma 6. *For any x and $y\in\Omega_0$, we have*

$$N_1(\lambda, x, y)-N_0(\lambda, x, y)=\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1\frac{d}{dt}N_t(\lambda, x, y)dt$$

$$N_0(\lambda, x, y)-N_{-1}(\lambda, x, y)=\lim_{\varepsilon\downarrow 0}\int_{-1}^{-\varepsilon}\frac{d}{dt}N_t(\lambda, x, y)dt.$$

Proof. These are direct consequences of Lemma 5 and the Hadamard variational formula.

Lemma 7. *For any $x\in\Omega_{-1}$, we have*

$$\int_{-1}^1\left|\frac{d}{dt}N_t(\lambda, x, x)\right|dx<\infty.$$

Proof. As a consequence of (1.4), we have the Hadamard variational inequality $(d/dt)N_t(\lambda, x, x)\geq 0$ for any $x\in\Omega_{-1}$ and $t\neq 0$. On the other hand, we have

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1\frac{d}{dt}N_t(\lambda, x, x)dt=N_1(\lambda, x, x)-N_0(\lambda, x, x)$$

$$\lim_{\varepsilon\downarrow 0}\int_{-1}^{-\varepsilon}\frac{d}{dt}N_t(\lambda, x, x)dt=N_0(\lambda, x, x)-N_{-1}(\lambda, x, x).$$

Lemma 7 follows from these.

Proof of Theorem. From (1.4), we have Hadamard's variational inequality:

$$\left|\frac{d}{dt}N_t(\lambda, x, y)\right| \leq \left[\frac{d}{dt}N_t(\lambda, x, x)\right]^{1/2}\left[\frac{d}{dt}N_t(\lambda, y, y)\right]^{1/2}$$

for $t\neq 0$. Thus for any $\varepsilon>0$, we have

$$\int_\varepsilon^1\left|\frac{d}{dt}N_t(\lambda, x, y)\right|dt$$
$$\leq\left[\int_\varepsilon^1\frac{d}{dt}N_t(\lambda, x, x)dt\right]^{1/2}\left[\int_\varepsilon^1\frac{d}{dt}N_t(\lambda, y, y)dt\right]^{1/2}.$$

This and Lemma 7 prove that

$$\int_0^1\left|\frac{d}{dt}N_t(\lambda, x, y)\right|dt<\infty.$$

Similarly, we have

$$\int_{-1}^0\left|\frac{d}{dt}N_t(\lambda, x, y)\right|dt<\infty.$$

These prove (1.6). (1.6) and Lemma 6 prove (1.7). The theorem has been proved.

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