76. Some Explicit Formulae in the Theory of Numbers

A Remark on the Riemann Hypothesis

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(Communicated by Kunihiko KODAIRA, M. J. A., June 11, 1981)

§1. **Introduction.** We put for $x \geqq 1$ and for $0<\alpha\leqq 1$,

$$H(x,\alpha)=\sum_{1\leqq n\leqq x}\sum_{1\leqq h\leqq \alpha n}\log (h,n)-\alpha\sum_{1\leqq n\leqq x}\sum_{1\leqq h\leqq n}\log (h,n)+\frac{1}{2}\sum_{n\leqq x}\log n,$$

where (h,n) is the greatest common divisor of h and n. We put for irrational α

$$F(x,\alpha)=H(x,\alpha)+xZ_\alpha(1)$$

and for rational $\alpha=a/q$ with $(a,q)=1$,

$$F(x,\alpha)=H(x,a/q)-(x/2q)\log (x/q)+x(\lambda_1(a/q)+\lambda_2(a/q)) \\ -(x/q)(\lambda_3(a/q)+\lambda_4(a/q)),$$

where we put

$$\lambda_j(a/q)=\sum_{b=1}^{q}\left(\left\{\frac{ab}{q}\right\}-\frac{1}{2}\right)\nu_j(b) \qquad \text{for } 1\leqq j\leqq 4,\ \nu_1(b)=1/b,$$

$\nu_2(b)=1/(b+q)$, $\nu_3(b)=\log (1+b/q)$ and $\nu_4(b)=2+\gamma_0-\gamma_{b/q}$ with

$$\gamma_\eta=-\int_1^\infty \frac{\{y\}dy}{(y+\eta)^2},$$

$\{y\}$ is the fractional part of y and $Z_\alpha(1)$ is defined below. Under these notations we have shown in [2] the following two theorems which are stated in a slightly different way.

Theorem 1. *The Riemann Hypothesis is equivalent to the statement that for any positive ε and for $X>X_0$,*

$$\int_0^1 |F(X,\alpha)|^2\,d\alpha \ll X^{1+\varepsilon}.$$

Theorem 2. *Let Q be an integer $\geqq 1$. Let $f_1, f_2, \cdots, f_i, \cdots, f_A$ be the Farey series of order Q, namely, $f_i=a_i/q_i$ with integral a_i and q_i, $(a_i, q_i)=1$, $0<a_i\leqq q_i$, $0<q_i\leqq Q$ and $f_1<f_2<\cdots<f_A$. Then the Riemann Hypothesis is equivalent to the statement that for any positive ε and for $Q>Q_0$,*

$$\sum_{i=1}^{A}|F(Q,a_i/q_i)|^2 \ll Q^{3+\varepsilon}.$$

In fact, the gap between above Theorem 1 and our previous Theorem 1 in [2] can be filled by the proof of Lemma 3 below and the gap between above Theorem 2 and our previous Theorem 2 in [2] will be filled in §2. The purpose of the present note is to give, by the classical methods, an explicit relation between $F(X,\alpha)$ for an individual

α and the totality of the non-trivial zeros of the Riemann zeta function $\zeta(s)$.

For this purpose we need some properties of the zeta function $Z_\alpha(s)$ defined by

$$Z_\alpha(s)=\sum_{n=1}^{\infty}\frac{\{\alpha n\}-1/2}{n^s}.$$

If α is rational and is $=a/q$, $(a, q)=1$, then

$$Z_\alpha(s)=q^{-s}\sum_{b=1}^{q}\left(\left\{\frac{ab}{q}\right\}-\frac{1}{2}\right)\zeta\left(s,\frac{b}{q}\right),$$

where $\zeta(s,w)$ for $0<w\leqq 1$ is the Hurwitz zeta function defined by $\zeta(s,w)=\sum_{n=0}^{\infty}(n+w)^{-s}$. Using the well known properties (cf. p. 37 of [9] and p. 114 and p. 115 of [8]) of $\zeta(s,w)$ we can show

Theorem 3. *Let $Q>Q_0$. Then*

$$\int_1^Q\left(\int_1^v F\left(y,\frac{a}{q}\right)\frac{dy}{y}\right)\frac{dv}{v}=\sum_\rho Z_{a/q}(\rho)\frac{Q^\rho}{\rho^3}+A_1\log^2 Q$$
$$+A_2\log Q+A_3+O(Q^{-1+\delta}),$$

where ρ runs over all non-trivial zeros of $\zeta(s)$, A_1, A_2 and A_3 are some constants independent of Q and δ is an arbitrary small positive number.

For irrational α, our knowledge of $Z_\alpha(s)$ seems to be scarce except Hecke's [4] for quadratic irrational α (cf. also Hardy and Littlewood [3]). Let D be a positive square free integer $\equiv 2$ or 3 (mod 4) and let η be the fundamental unit of the quadratic number field $Q(\sqrt{D})$ or the square of it as in [4]. Then as a simple application of Hecke's work [4], we can show

Theorem 4. *Let $X>X_0$. Then*

$$\int_1^X\left(\int_1^v F(y,1/\sqrt{D})\frac{dy}{y}\right)\frac{dv}{v}=\sum_\rho Z_{1/\sqrt{D}}(\rho)\frac{X^\rho}{\rho^3}+\sum_{n=-\infty}^{+\infty}C_n X^{\frac{2\pi i n}{\log\eta}}$$
$$+A_1\log^3 X+A_2\log^2 X+A_3\log X+O(X^{-1+\delta}),$$

where δ is an arbitrary small positive number, A_1, A_2, A_3 and C_n are some constants independent of X and $C_n\ll n^{-2+\delta}$.

We shall prove Theorem 2 in § 2 and Theorem 4 in § 3. Since the proof of Theorem 3 is similar to that of Theorem 4, we shall omit it. We always denote positive absolute constants by C and arbitrarily small positive numbers by ε and write $s=\sigma+it$.

§ 2. Proof of Theorem 2. It is enough to prove the following two lemmas. Let $0<a\leqq q\leqq Q$, $(a, q)=1$ and $Q>Q_0$.

Lemma 1.

$$Q\sum_{n\leqq Q}\frac{\{an/q\}-1/2}{n}=-\frac{1}{2}\frac{Q}{q}\log(Q/q)+Q\left(\lambda_1\left(\frac{a}{q}\right)+\lambda_2\left(\frac{a}{q}\right)\right)$$
$$-\frac{Q}{q}\left(\lambda_3\left(\frac{a}{q}\right)-\sum_{b=1}^{q}\left(\left\{\frac{ab}{q}\right\}-\frac{1}{2}\right)\gamma_{b/q}\right)+S\left(\frac{a}{q}\right)-\tilde{S}\left(\frac{a}{q}\right),$$

where we put

$$S\left(\frac{a}{q}\right)=\frac{Q}{q}\sum_{b=1}^{q}\left(\left\{\frac{ab}{q}\right\}-\frac{1}{2}\right)I \quad with \quad I=\int_{Q/q}^{\infty}\frac{\{y-(b/q)\}-(1/2)}{y^2}dy$$

and

$$\tilde{S}\left(\frac{a}{q}\right)=\sum_{b=1}^{q}\left(\left\{\frac{ab}{q}\right\}-\frac{1}{2}\right)\left(\left\{\frac{Q-b}{q}\right\}-\frac{1}{2}\right).$$

Proof. Since the left hand side is

$$=Q\left(\lambda_1\left(\frac{a}{q}\right)+\lambda_2\left(\frac{a}{q}\right)\right)+\frac{Q}{q}\sum_{b=1}^{q}\left(\left\{\frac{ab}{q}\right\}-\frac{1}{2}\right)\sum_{1<m\leqq(Q-b)/q}\left(m+\frac{b}{q}\right)^{-1}$$

and

$$\sum_{1<m\leqq(Q-b)/q}\left(m+\frac{b}{q}\right)^{-1}=\log\frac{Q}{q}-\log\left(1+\frac{b}{q}\right)+\gamma_{b/q}+I-\frac{q}{Q}\left(\left\{\frac{Q-b}{q}\right\}-\frac{1}{2}\right),$$

we get our Lemma 1. Q.E.D.

Lemma 2. *$\sum_{q\leqq Q}\sum_a'|S(a/q)|^2\ll Q^{3+\varepsilon}$ and $\sum_{q\leqq Q}\sum_a'|\tilde{S}(a/q)|^2\ll Q^{3+\varepsilon}$, where the dash indicates that we sum over all a in $1\leqq a\leqq q$ with $(a,q)=1$.*

Proof. We denote the sum $\sum_{1\leqq n\leqq q-1}$ by $\sum_n''$.

$$S\left(\frac{a}{q}\right)=\frac{Q}{q}\sum_b''\left(\sum_m''\frac{\sin(2\pi mab/q)}{m\pi}+O\left(\left(q\left\|\frac{ab}{q}\right\|\right)^{-1}\right)\right)$$
$$\times\left(\sum_k''\frac{\mathrm{Im}\,(e(-kb/q)I(k))}{k\pi}+O((q(Q/q)^2)^{-1})\right)+O(1)$$
$$\ll\frac{Q}{q}\sum_k''\sum_m''\frac{|I(k)|}{mk}\left(\left|\sum_b'' e\left(\frac{b}{q}(k+ma)\right)\right|+\left|\sum_b'' e\left(\frac{b}{q}(k-ma)\right)\right|\right)$$
$$+\frac{q\log q}{Q}+1$$
$$\ll q\sum_k''\sum_{\substack{m\\ q|k+ma}}''(k^2m)^{-1}+q\sum_k''\sum_{\substack{m\\ q|k-ma}}''(k^2m)^{-1}$$
$$+\sum_k''\sum_{\substack{m\\ q\nmid k+ma}}''\left(k^2m\left\|\frac{k+ma}{q}\right\|\right)^{-1}+\sum_k''\sum_{\substack{m\\ q\nmid k-ma}}''\left(k^2m\left\|\frac{k-ma}{q}\right\|\right)^{-1}$$
$$+\log q$$
$$=S_1\left(\frac{a}{q}\right)+S_2\left(\frac{a}{q}\right)+S_3\left(\frac{a}{q}\right)+S_4\left(\frac{a}{q}\right)+\log q,$$

say, where

$$I(k)=\int_{Q/q}^{\infty}\frac{e(ky)}{y^2}dy\ll(k(Q/q)^2)^{-1},\qquad e(x)=\exp(2\pi ix),$$

$\|x\|=\mathrm{Min}(\{x\},1-\{x\})$ and we have used the expression

$$\{y\}-\frac{1}{2}=\sum_{k=1}^{\infty}\frac{\sin(2k\pi y)}{k\pi}$$

if y is not an integer.

$$\sum_{q\leqq Q}\sum_a'\left|S_1\left(\frac{a}{q}\right)\right|^2\ll\sum_{q\leqq Q}q^2\log q\sum_a'\sum_k''\sum_{\substack{m\\ q|k+ma}}''(k^2m)^{-1}$$

$$\ll \sum_{q\leqq Q} q^2 \log q \sum_{d|q} \sum_{\substack{m \\ (m,q)=d}}'' \frac{1}{m} \sum_{k}'' \frac{1}{k^2} \sum_{\substack{a \\ q|k+ma}}' \cdot 1$$

$$\ll \sum_{q\leqq Q} q^2 \log q \sum_{d|q} \sum_{d|m}'' \sum_{d|k}'' dk^{-2}m^{-1} \ll Q^3 \log^2 Q.$$

Similarly, we get the same upper bound for the sum of $S_2(a/q)$'s.

$$\sum_{q\leqq Q} \sum_{a}' \left|S_3\left(\frac{a}{q}\right)\right|^2 \ll \sum_{q\leqq Q} \log q \sum_{a}' \sum_{\substack{k \\ q\nmid k+ma}}'' \sum_{m}'' \left(k^2 m \left\|\frac{k+ma}{q}\right\|^2\right)^{-1}$$

$$\ll \sum_{q\leqq Q} \log q \sum_{c}'' \|c/q\|^{-2} \sum_{d|q} \sum_{\substack{m \\ (m,q)=d}}'' m^{-1} \sum_{k}'' k^{-2} \sum_{k+ma\equiv c \pmod q}' \cdot 1$$

$$\ll \sum_{q\leqq Q} q^2 \log^2 q \left(\sum_{d|q} 1\right) \ll Q^3 \log^3 Q.$$

Similarly, we get the same upper bound for the sum of $S_4(a/q)$'s. Thus we get $\sum_{q\leqq Q} \sum_a' |S(a/q)|^2 \ll Q^3 \log^3 Q$. In the same manner, we get $\sum_{q\leqq Q} \sum_a' |\tilde{S}(a/q)|^2 \ll Q^3 \log^5 Q$. Q.E.D.

§3. Proof of Theorem 4. We suppose first that α is irrational and remark the following lemma and its corollary.

Lemma 3. *For almost all irrational α, $Z_\alpha(s)$ is regular in* Re $s>0$ *and $Z_\alpha(s) \ll (\log T)^{2+\varepsilon}$ for $\sigma \geqq 1-C/\log T$, $|t|\leqq T$ and $T>T_0$.*

Proof. We remark that $\sum_{n\leqq y} (\{\alpha n\}-1/2) \ll (\log y)^{2+\varepsilon}$ for $y>y_0$ and for almost all irrational α (cf. p. 38 of Lang [7]). Now let α satisfy this inequality and N be an integer $\geqq 1$. Then for Re $s>1$

$$Z_\alpha(s) = \sum_{n<N} \frac{\{\alpha n\}-1/2}{n^s} + s\int_N^\infty \frac{\sum_{N\leqq n\leqq y} (\{\alpha n\}-1/2)}{y^{s+1}} dy.$$

Hence $Z_\alpha(s)$ is regular for Re $s>0$ and the rest can be proved in the same way as p. 114 of [8]. Q.E.D.

Corollary. *For almost all irrational α,*

$$F(X,\alpha) = O(X \exp(-C\sqrt{\log X})).$$

Since $H(X,\alpha) = -\sum_{dm\leqq X} \Lambda(d)(\{\alpha m\}-1/2)$, we get the above corollary, as usual, by the contour integral of $(\zeta'/\zeta)(s) Z_\alpha(s) X^s/s$ using Lemma 3 and p. 69 of [8], where $\Lambda(d)$ is the von Mangoldt function.

Now we shall prove our Theorem 4 and suppose that $\alpha = 1/\sqrt{D}$ as in §1. We need the following lemma due to Hecke [4].

Lemma 4. i) *$Z_{1/\sqrt{D}}(s)$ is regular for* Re $s>0$ *and in* Re $s\leqq 0$ *has only simple poles at most at the points*

$$s = -2n \pm \frac{2\pi i k}{\log \eta}, \qquad n, k = 0, 1, 2, \cdots.$$

ii) *$H(s) Z_{1/\sqrt{D}}(s) \ll |t|^{1-\sigma+\varepsilon}$ for $-1\leqq\sigma\leqq 1$, where $H(s) = \prod_{n=0}^\infty (1-\eta^{-s-2n})$.*

Now we consider the integral

$$I = \int_1^X \left(\int_1^v F(y,\alpha)\frac{dy}{y}\right)\frac{dv}{v}.$$

$$I = Z_\alpha(1)(X-1-\log X) - \frac{1}{2}\sum_{n\leqq X}\left(\sum_{dm=n} \Lambda(d)\left(\{\alpha m\}-\frac{1}{2}\right)\right)\left(\log\frac{X}{n}\right)^2.$$

Here we remark that for any integral $k>k_0$, we can take T_k such that

$$\frac{2\pi k}{\log \eta} < T_k < \frac{2\pi(k+1)}{\log \eta}, \qquad \frac{\zeta'}{\zeta}\left(\sigma \pm iT_k\right) \ll \log^2 T_k$$

and $H(\sigma \pm iT_k)^{-1} \ll 1$ for $-1 \leqq \sigma \leqq 2$. With this T_k we have first

$$I = Z_\alpha(1)(X-1-\log X) + \frac{1}{2\pi i}\int_{2-iT_k}^{2+iT_k} \frac{\zeta'}{\zeta}(s) Z_\alpha(s) \frac{X^s}{s^3}\, ds + O\left(\frac{X^2}{T_k^2}\right).$$

Next, we move the line of the integration to $(-1+\delta-iT_k, -1+\delta+iT_k)$ for any small positive $\delta < 1$. Then

$$\begin{aligned} I = {} & Z_\alpha(1)(X-1-\log X) + \sum_{|\mathrm{Im}\,\rho| < T_k} Z_\alpha(\rho)\frac{X^\rho}{\rho^3} + \sum_{n=-k}^{k} C_n X^{2\pi i n/\log \eta} \\ & - XZ_\alpha(1) + A_1 \log^3 X + A_2 \log^2 X + A_3 \log X + O(X^2 T_k^{-2}) \\ & + O\left(T_k^{-3}\log^2 T_k \int_{-1+\delta}^{2} X^\sigma |Z_\alpha(\sigma \pm iT_k)|\, d\sigma\right) \\ & + O\left(X^{-1+\delta}\int_{-T_k}^{T_k} \left|\frac{\zeta'}{\zeta}(-1+\delta+it)\right| |Z_\alpha(-1+\delta+it)|\,|-1+\delta+it|^{-3} dt\right). \end{aligned}$$

The last two terms are $\ll X^2(\log^2 T_k) T_k^{-(1+\delta/2)} + X^{-1+\delta}$. Letting k tend to ∞, we get our Theorem 4.

References

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