# 74. *On Eisenstein Series for Siegel Modular Groups. II*

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**Introduction.** This is a continuation of [11]. We note the Fourier coefficients of Eisenstein series concerning the following two points: the rationality and an interpretation of the explicit formula. The author would like to thank Profs. M. Harris, T. Oda, and D. Zagier for kindly communicating their preprints: [4], [5], [18], [20]. (This paper was revised into the present form in March-April 1981 when the author received their preprints; the original preprint was cited in [9-II], [12].) We follow the previous notations of [8]–[12].

§1. ***L*-functions.** We fix our notation on two $L$-functions attached to Siegel eigenforms. Let $f$ be a Siegel eigen modular form in $M_k(\Gamma_n)$ for integers $n\geqq 0$ and $k\geqq 0$. We denote by $L_1(s, f)$ the first $L$-function attached to $f$ (an Euler product over $\boldsymbol{Q}$ of degree $2^n$), which is defined in Andrianov [1]. We put $L_1^u(s, f)=L_1(s+n(2k-n-1)/4, f)$. If $n=0$ then $L_1(s, f)=L_1^u(s, f)=\zeta(s)$ (cf. [10, §3]). It is expected that $L_1^u(s, f)$ is meromorphic on $\boldsymbol{C}$ with functional equation for $s\to 1-s$. This is known for $n\leqq 2$. A formulation of Ramanujan conjecture for $f$ is that $L_1^u(s, f)$ is unitary in the sense of [7]; cf. [8, p. 150, p. 165]. We denote by $L_2^u(s, f)$ the second $L$-function attached to $f$ (an Euler product over $\boldsymbol{Q}$ of degree $2n+1$), which is defined in Andrianov [2]. If $n=0$ then $L_2^u(s, f)=\zeta(s)$. It is expected that $L_2^u(s, f)$ is meromorphic on $\boldsymbol{C}$ with functional equation for $s\to 1-s$. This is proved in certain cases by Shimura [19] and Andrianov-Kalinin [3]. For $n=1$ we have $L_2^u(s, f)=L_2(s+k-1, f)$ in the previous notation.

We note relations between $L$-functions for two liftings.

(A) For each eigen modular form $f$ in $M_k(\Gamma_1)$ we have

$$L_1^u(s, [f])=L_1^u(s+(k-2)/2, f)L_1^u(s-(k-2)/2, f) \quad \text{and}$$
$$L_2^u(s, [f])=L_2^u(s, f)\zeta(s+k-2)\zeta(s-k+2).$$

More generally let $F$ be an eigen modular form in $M_k(\Gamma_n)$ such that $\Phi(F)\neq 0$. Then we have

$$L_1^u(s, F)=L_1^u(s+(k-n)/2, \Phi(F))L_1^u(s-(k-n)/2, \Phi(F)) \quad \text{and}$$
$$L_2^u(s, F)=L_2^u(s, \Phi(F))\zeta(s+k-n)\zeta(s-k+n).$$

(B) For each eigen modular form $f$ in $M_{2k-2}(\Gamma_1)$ we have

$$L_1^u(s, \sigma_k(f))=L_1^u(s, f)\zeta(s+1/2)\zeta(s-1/2) \quad \text{and}$$
$$L_2^u(s, \sigma_k(f))=L_1^u(s+1/2, f)L_1^u(s-1/2, f)\zeta(s).$$

§2. **Fourier coefficients.** For a modular form $f$ in $M_k(\Gamma_n)$ ($n\geqq 0$

and $k \geqq 0$) we denote by $\boldsymbol{Q}(f)^*$ the extension field of $\boldsymbol{Q}$ generated by the Fourier coefficients $a(T, f)$ for all $T \geqq 0$. If $f$ is an eigen modular form, then $\boldsymbol{Q}(f)^*$ contains the field $\boldsymbol{Q}(f)=\boldsymbol{Q}(\lambda(f))$ defined in [10]. If the multiplicity of the eigencharacter $\lambda(f)$ is one (i.e., $m(\lambda(f))=1$) and $k$ is even, then there exists a non-zero constant $\gamma \in \boldsymbol{C}$ such that $\boldsymbol{Q}(\gamma f)^*=\boldsymbol{Q}(f)=\boldsymbol{Q}(\gamma f)$ by Theorem 3 of [10].

We first treat Eisenstein series attached to the space $M_k^{\mathrm{I}}(\Gamma_2)$ (in the notation of [9-II]). There exists a bijection $\sigma_k : M_{2k-2}(\Gamma_1) \to M_k^{\mathrm{I}}(\Gamma_2)$ constructed by Zagier [20] using the result of Kohnen [6]. The following theorem is proved similarly as in [11, Theorems 2–4].

**Theorem 1.** *Let $f$ be a modular form in $M_k^{\mathrm{I}}(\Gamma_2)$ for an even integer $k>n+2$ with an integer $n \geqq 2$. Then:*

(1) *Assume that $f$ is an eigen modular form. Then*

$$m(\lambda([f]^{(n-2)}))=1.$$

(2) *Assume that $f$ is an eigen modular form, and let $F$ be an eigen modular form in $M_k(\Gamma_n)$ satisfying $\Phi^{n-2}(F)=f$. Then $F=[f]^{(n-2)}$.*

(3) *For each $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ we have $\sigma([f]^{(n-2)})=[\sigma(f)]^{(n-2)}$.*

(4) $\boldsymbol{Q}([f]^{(n-2)})^*=\boldsymbol{Q}(f)^*$.

(5) *Assume that $f$ is an eigen modular form. Then there exists a non-zero constant $\gamma \in \boldsymbol{C}$ such that all the Fourier coefficients of $\gamma[f]^{(n-2)}$ belong to $\boldsymbol{Z}(f)$.*

**Proof.** The properties (2)–(5) follow from (1) as in [11]. (Note that Theorem 3(2) of [11] is equivalent to $\boldsymbol{Q}([f]^{(n-1)})^*=\boldsymbol{Q}(f)^*$.) To prove (1) let $F$ be an eigen modular form in $M_k(\Gamma_n)$ satisfying $\lambda(F)=\lambda([f]^{(n-2)})$. Put $g=\Phi^{n-2}(F)$, then we have $g \neq 0$ as in the proof of Theorem 2 in [11] by using $\lambda(p, f)=p^{k-2}+p^{k-1}+\lambda(p, f_1)$ with an eigen modular form $f_1$ in $M_{2k-2}(\Gamma_1)$. (We remark that, in [11] p. 52, l. 4 from the bottom, "$r$" should be "$n-r$". For another method, see the proof of Theorem 2 below.) Hence $g$ is an eigen modular form in $M_k(\Gamma_2)$ satisfying $\lambda(g)=\lambda(f)$ and $L_1(s, g)=L_1(s, f)=\zeta(s-k+2)\zeta(s-k+1)L_1(s, f_1)$, so $L_1(s, g)$ has a simple pole at $s=k$ of residue $\zeta(2)L_1(k, f_1)>0$ (cf. [8, 2.2(2)]). Hence we have $g \in M_k^{\mathrm{I}}(\Gamma_2)$, since if $g \in M_k^{\mathrm{II}}(\Gamma_2)$ then $L_1(s, g)$ is holomorphic on $\boldsymbol{C}$ by Oda [18] proved by using his results [17]. (This holomorphy supports the Ramanujan conjecture [8, Conjecture 3]. We may also use the result of Evdokimov cited in Oda [18] and Zagier [20].) By the multiplicity one theorem for $M_{2k-2}(\Gamma_1)$ and the bijection $\sigma_k$, there exists a non-zero constant $\gamma \in \boldsymbol{C}$ such that $g=\gamma f$. We put $H=F-\gamma[f]^{(n-2)}$. If $H \neq 0$, then $H$ is an eigen modular form in $M_k(\Gamma_n)$ satisfying $\lambda(H)=\lambda(F)=\lambda([f]^{(n-2)})$ and $\Phi^{n-2}(H)=0$. This is impossible as above. Hence $H=0$, and we have $F=\gamma[f]^{(n-2)}$. Q.E.D.

Next, we treat a more general case. The properties (2)–(4) of Theorem 1 are generalized as in Theorem 2 below. Similar results are obtained in Harris [5].

**Theorem 2.** *Let $f$ be a modular form in $M_k(\Gamma_r)$ for $r \geqq 0$ and even $k > n+r+1$ with $n \geqq r$. Then:*

(1) *Assume that $f$ is an eigen modular form. Let $F$ be an eigen modular form in $M_k(\Gamma_n)$ satisfying $\Phi^{n-r}(F)=f$. Then $F=[f]^{(n-r)}$.*

(2) *For each $\sigma \in \mathrm{Aut}\,(\boldsymbol{C})$ we have $\sigma([f]^{(n-r)})=[\sigma(f)]^{(n-r)}$.*

(3) $\boldsymbol{Q}([f]^{(n-r)})^* = \boldsymbol{Q}(f)^*$.

**Proof.** We prove (1); (2), (3) follow from (1) as in [11, Theorem 3]. We show first that $\Phi^{n-r}(F) \neq 0$. Suppose that $\Phi^{n-r}(F)=0$ and let $n-j$ be the maximal integer $\leqq n$ such that $\Phi^{n-j}(F) \neq 0$. Then $r+1 \leqq j \leqq n$ and $h=\Phi^{n-j}(F)$ is an eigen cusp form of degree $j$. Let $r-r(0)$ be the maximal integer $\leqq r$ such that $\Phi^{r-r(0)}(f) \neq 0$, and put $f_0=\Phi^{r-r(0)}(f)$. Then $f_0$ is an eigen cusp form of degree $r(0)$. Concerning the second $L$-function, as in § 1, we have:

$$L_2^u(s,F)=L_2^u(s,h)\prod_{i=j+1}^{n}\zeta(s+k-i)\zeta(s-k+i),$$

$$L_2^u(s,[f]^{(n-r)})=L_2^u(s,f_0)\prod_{i=r(0)+1}^{n}\zeta(s+k-i)\zeta(s-k+i).$$

Hence, by using $\lambda(F)=\lambda([f]^{(n-r)})$ (cf. [11, Remark 1]), we have

$$(*)\qquad L_2^u(s,h)=L_2^u(s,f_0)\prod_{i=r(0)+1}^{j}\zeta(s+k-i)\zeta(s-k+i).$$

Since the Euler product $L_2^u(s,h)$ converges absolutely in $\mathrm{Re}\,(s) > j+1$ (see Andrianov-Kalinin [3]), the left hand side of $(*)$ is holomorphic at $s=k-r(0) > j+1$, but the right hand side of $(*)$ has a simple pole at $s=k-r(0)$ coming from the simple pole of $\zeta(s-k+r(0)+1)$. This contradiction shows that $\Phi^{n-r}(F) \neq 0$. We put $H=F-[f]^{(n-r)}$. If $H \neq 0$, then $H$ is an eigen modular form in $M_k(\Gamma_n)$ satisfying $\lambda(H)=\lambda(F)=\lambda([f]^{(n-r)})$ and $\Phi^{n-r}(H)=0$. This is impossible as shown above. Hence $H=0$, and we have $F=[f]^{(n-r)}$. Q.E.D.

Assuming a multiplicity one condition, we have Theorem 3 below. This assumption is satisfied, for example, if $f$ belongs to $M_k(\Gamma_1)$ (resp. $M_k^{\mathrm{I}}(\Gamma_2)$), where Theorem 3 corresponds to [11, Theorems 2 and 4] (resp. Theorem 1 above).

**Theorem 3.** *Let $f$ be an eigen modular form in $M_k(\Gamma_r)$ for $r \geqq 0$ and even $k > n+r+1$ with $n \geqq r$. Assume that $m(\lambda(f))=1$. Then:*

(1) $m(\lambda([f]^{(n-r)}))=1$.

(2) $\boldsymbol{Q}([f]^{(n-r)})=\boldsymbol{Q}(f)$.

(3) *There exists a non-zero constant $\gamma \in \boldsymbol{C}$ such that all the Fourier coefficients of $\gamma[f]^{(n-r)}$ belong to $\boldsymbol{Z}(f)$. In particular $\boldsymbol{Q}(\gamma[f]^{(n-r)})^* = \boldsymbol{Q}(\gamma f)^* = \boldsymbol{Q}(f) = \boldsymbol{Q}([f]^{(n-r)})$.*

**Proof.** Let $F$ be an eigen modular form in $M_k(\Gamma_n)$ satisfying $\lambda(F)=\lambda([f]^{(n-r)})$. As shown in the proof of Theorem 2 above, we have $\Phi^{n-r}(F) \neq 0$. Put $g=\Phi^{n-r}(F)$, then $g$ is an eigen modular form in $M_k(\Gamma_r)$

satisfying $\lambda(g)=\lambda(f)$. Hence, by the assumption $m(\lambda(f))=1$, there exists a non-zero constant $\gamma\in C$ such that $g=\gamma f$. Put $H=F-\gamma[f]^{(n-r)}$. If $H\neq 0$, then $H$ is an eigen modular form in $M_k(\Gamma_n)$ satisfying $\lambda(H)=\lambda(F)=\lambda([f]^{(n-r)})$ and $\Phi^{n-r}(H)=0$. This is impossible as above. Hence $H=0$, and $F=\gamma[f]^{(n-r)}$. This proves (1). From $m(\lambda(f))=1$ (resp. $m(\lambda(F))=1$) and Theorem 3 of [10], there exists a non-zero constant $\gamma_1$ (resp. $\gamma_2$) in $C$ such that $\boldsymbol{Q}(\gamma_1 f)^*=\boldsymbol{Q}(f)$ (resp. $\boldsymbol{Q}(\gamma_2 F)^*=\boldsymbol{Q}(F)$). Hence, by using Theorem 2(3) above, we have $\boldsymbol{Q}(F)\subset\boldsymbol{Q}(\gamma_1 F)^*=\boldsymbol{Q}(\gamma_1 f)^*=\boldsymbol{Q}(f)$ and $\boldsymbol{Q}(f)\subset\boldsymbol{Q}(\gamma_2 f)^*=\boldsymbol{Q}(\gamma_2 F)^*=\boldsymbol{Q}(F)$. Hence $\boldsymbol{Q}(F)=\boldsymbol{Q}(f)$. This proves (2). Now (3) follows from (1) and (2) by applying Theorem 3 of [10].

Q.E.D.

**§3. An interpretation of the explicit formula.** For each eigen modular form $f$ in $M_k(\Gamma_1)$ we have the explicit formula of the Fourier coefficients $a(T,[f])$ of $[f]$ in [12, Theorem 3]; see Maass [13], [14] and Mizumoto [16]. Here we note an interpretation of this formula.

(1) Let $f$ and $g$ be modular forms of weight $k$ and $l$ respectively for congruence subgroups of the Siegel modular group of degree $n\geqq 1$. We assume that they have the Fourier expansions of the following form: $f=\sum_N a(N)q^N$ and $g=\sum_N b(N)q^N$, where $N$ runs over all $n\times n$ symmetric semi-integral positive semi-definite matrices, and $q^N=\exp(2\pi\sqrt{-1}\cdot\mathrm{trace}\,(NZ))$ with a variable $Z$ on the Siegel upper half space of degree $n$. We put $D(s,f,g)=\sum_{\{N\}} a(N)b(N)\varepsilon(N)^{-1}\det(N)^{-s}$, where $s$ is a variable in $C$, $\{N\}$ runs over all unimodular $(GL(n,\boldsymbol{Z})$-) equivalence classes of all $N>0$, and $\varepsilon(N)=\sharp\{U\in GL(n,\boldsymbol{Z})\,|\,{}^tUNU=N\}$. This Dirichlet series ("Rankin convolution") was studied by Maass [15]. It is expected that $D(s,f,g)$ is meromorphic on $C$. To have a simple functional equation it would be better to consider the normalization $D^*(s,f,g)$ which would be obtained from $D(s,f,g)$ by multiplying a finite number of Dirichlet $L$-functions. (Such a normalization is known in certain cases.) Then $D^*(s,f,g)$ would have a functional equation for $s\to k+l-(n+1)/2-s$.

(2) Let $n\geqq 1$ and $r\geqq 1$ be integers. Let $T$ be an $n\times n$ symmetric semi-integral positive definite matrix. We denote by $\vartheta_T^{(r)}$ the usual theta function of degree $r$ attached to $T$:

$$\vartheta_T^{(r)}=\sum_M \exp(2\pi\sqrt{-1}\cdot\mathrm{trace}\,({}^tMTMZ)),$$

where $M$ runs over all $n\times r$ integral matrices and $Z$ is a variable on the Siegel upper half space of degree $r$. It is known that $\vartheta_T^{(r)}$ is a modular form of weight $n/2$ for a certain congruence subgroup of the Siegel modular group of degree $r$. Let $f$ be an eigen modular form in $M_k(\Gamma_r)$ for $k\geqq 0$. For $s_1$ and $s_2$ in $C$, we put

$$b^{(*)}(T,f;s_1,s_2)=D^{(*)}(s_1,f,\vartheta_T^{(r)})/L_2^u(s_2,f),$$

where $b^{(*)}$ (resp. $D^{(*)}$) indicates $b$ or $b^*$ (resp. $D$ or $D^*$). (We note that

$b^{(*)}=D^{(*)}=0$ if $n<r$.) To be precise, here we assume that $\varphi(s)=D^{(*)}(s_1+s, f, \vartheta_T^{(r)})/L_2^u(s_2+s, f)$ is holomorphic as a function of $s$ at $s=0$, and we understand that $b^{(*)}(T, f; s_1, s_2)=\varphi(0)$. We put $b^{(*)}(T, f)=b^{(*)}(T, f; k-1, k-1)=D^{(*)}(k-1, f, \vartheta_T^{(r)})/L_2^u(k-1, f)$.

(3) The explicit formula would suggest the following. Let $F$ be an eigen modular form in $M_k(\Gamma_n)$ for $n\geqq 1$ and $k\geqq 0$. Let $T$ be an $n\times n$ symmetric positive definite matrix. *Then we might have "$a(T, F)=b^{(*)}(T, \Phi^{n-r}(F))$" up to elementary factors for each $r$ in $1\leqq r\leqq n$ such that $\Phi^{n-r}(F)\neq 0$.* There exist two supporting examples: (I) $n=2$ and $r=1$, (II) $n=r\geqq 1$. The case (I) corresponds to the explicit formula: [12, Theorem 3], Maass [13], [14], and Mizumoto [16]. The case (II) corresponds to the results of Shimura [19] and Andrianov [2]. (In both cases $b^*(T, \Phi^{n-r}(F))$ is explicitly determined.)

(4) If $L(s)=\sum_{j\geqq r} c_j(s-s_0)^j$ is the Laurent expansion of a function $L(s)$ which is meromorphic at $s=s_0$ with $c_r\neq 0$, then we consider $V=c_r$ as the special value of $L(s)$ at $s=s_0$. Some results seem to suggest that we would have the expression $V=A\cdot P\cdot R$ for certain special values of $L$-functions, where $A$ is the "algebraic part", $P$ is the "period", and $R$ is the "regulator". Let $f$ be an eigen cusp form in $M_k(\Gamma_n)$. If $n=1$, then the special value $V=L_2^u(k-1, f)=L_2(2k-2, f)$ is written in the form $V=A\cdot P$. (We consider that $R=1$ here.) We refer to [9], [12] for an interpretation of the "numerator" of the "algebraic part" $A$ in connection with congruences and the explicit formula. If $n>1$, then we would need the "regulator" $R\neq 1$ for the special value of $L_2^u(s, f)$ at $s=k-1$. (Special values at certain $s<k-n+1$ are treated in Harris [4].) For example, does the interpretation "$a(T, F)=b^{(*)}(T, \Phi(F))$" for $F=[\chi_{10}]$ and $T>0$ of size 3 suggest that 53 might appear in the "numerator" of the "algebraic part" of the special value $L_2^u(9, \chi_{10})=L_1(18, \Delta_{18})L_1(17, \Delta_{18})\zeta(9)$ (or $L_1(18, \Delta_{18})$)?

(5) Let $f$ and $g$ be as in (1). Then the above interpretation in (3) is considered as an example of the expression of the special values of $D(s, f, g)$ by using the special values of $D(s, \Phi^{n-r}(f), \Phi^{n-r}(g))$ for $1\leqq r\leqq n$. (Note that $\vartheta_T^{(r)}=\Phi^{n-r}(\vartheta_T^{(n)})$ for theta functions in (2).) For example, for each eigen modular form $f$ in $M_k(\Gamma_1)$ and $T>0$ of size 2, we have "$b(T, [f])=b(T, f)$" from the above (I) and (II) ($n=r=2$), hence we have such an expression of the residue of $D(s, [f], \vartheta_T^{(2)})$ at $s=k-1$ (simple pole) by using $D(k-1, f, \vartheta_T^{(1)})$.

**Remark.** We have similar results for some other liftings containing generalizations and applications for the Eisenstein series map in the vector valued Hilbert-Siegel modular case.

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