71. Non-Commutative Lorentz Spaces Associated with a Semi-Finite von Neumann Algebra and Applications

By Hideki KOSAKI
Department of Mathematics, University of Kansas,
Lawrence, Kansas 66045 U.S.A.

(Communicated by Kôsaku YOSIDA, M. J. A., June 11, 1981)

§0. Introduction. The purpose of the note is to announce a construction, basic properties, and certain applications of non-commutative Lorentz spaces associated with semi-finite von Neumann algebras.

At first, non-increasing rearrangements for measurable *operators* affiliated with a *semi-finite* von Neumann algebra are introduced and basic properties are obtained. Then, based on them, non-commutative Lorentz spaces are defined. Since we show that those Lorentz spaces are identified with real interpolation spaces between the semi-finite von Neumann algebra in question and its predual, the abstract Marcinkiewicz theorem is available to our Lorentz spaces. In the last section, we obtain certain applications of the abstract Marcinkiewicz theorem.

Full details and further applications will be published elsewhere.

§1. Non-increasing rearrangements of measurable operators. Throughout the paper, $\mathcal{M}$ stands for a *semi-finite von Neumann algebra* with a *faithful* normal semi-finite *trace* τ. Also, by an operator, we will always mean a *τ-measurable* operator (affiliated with $\mathcal{M}$) in the sense of Segal [6]. For operators x, y, their strong sum and product are denoted by $x+y$, xy respectively. In other words, we shall omit closure signs, which will never make confusion [6].

Definition 1.1. Let x be an operator with the polar decomposition $u|x|$ and $|x|=\int_0^\infty s\,de_s$ be the spectral decomposition of $|x|$. For each $t\geq 0$, we set $\lambda_t(x)=\tau(1-e_t)$, the *distribution function* of x. Also, for each $t\geq 0$, we set $\mu_t(x)=\mathrm{Min}\,\{s\geq 0\,;\,\lambda_s(x)\geq t\}$, the *non-increasing rearrangement* of x.

This $\mu_t(x)$ was introduced by Fack [2] for *x in $\mathcal{M}$* as a generalization of the "t-th" largest eigenvalue of a compact operator (see [7]). We notice that, for any *operator* $x, \mu_t(x)<\infty$, $t>0$ (see [5]) and $\mu_0(x)=\|x\|$ if $x\in\mathcal{M}$. We also notice that, in the definition of $\mu_t(x)$, the minimum is actually attained due to the normality of τ.

The following result gives us characterizations of $\mu_t(x)$:

Theorem 1.2. *For an operator x and each $t>0$,*

$$\mu_t(x)=\inf\{\|xp\|;\ p \text{ is a projection in } \mathcal{M} \text{ and } \tau(1-p)\le t\}$$
$$=\inf\{\|x-y\|;\ y \text{ is an operator and } \tau\,(\text{supp. } y)\le t\}.$$

Here, supp. y *means the support of* $|y|=\sqrt{y^*y}$.

Corollary 1.3. *For operators x, y,*

(i) $\mu_{t+s}(x+y)\le\mu_t(x)+\mu_s(y)$; $t, s>0$,

(ii) $\mu_t(x)=\mu_t(x^*)=\mu_t(|x|)$, $t>0$.

(iii) $\mu_t(yx)\le\|y\|\,\mu_t(x)$, $t>0$ (if $y\in\mathcal{M}$).

The next result is useful because it allows us to use approximation argument.

Theorem 1.4. *Let $\{x_n\}$ be an increasing sequence of positive operators converging to x in the measure topology* [5]. *Then, for each $t>0$, $\mu_t(x_n)\uparrow\mu_t(x)$ as $n\to\infty$.*

The next result is known for x in $\mathcal{M}$ ([2]). Thus, the above theorem, the normality of τ, and the monotone convergence theorem yield:

Corollary 1.5. *For a positive operator x and a continuous increasing function f on* $[0,\infty)$ *with* $f(0)=0$, *we have*

$$\tau(f(x))=\int_0^\infty f(\mu_s(x))ds.$$

§ 2. Non-commutative Lorentz spaces and the real interpolation method. As a generalization of the usual Lorentz spaces (see [8]), we define our Lorentz spaces as follows:

Definition 2.1. For $1\le p<\infty$, $1\le q\le\infty$, let $Lpq\,(\mathcal{M};\tau)$ be the set of every (τ-measurable) operator x satisfying

$$\|x\|_{pq}^*=\left[\int_0^\infty \{t^{1/p}\mu_t(x)\}^q\frac{dt}{t}\right]^{1/q}<\infty.$$

As usual, $\|x\|_{p\infty}^*$ should be understood as

$$\|x\|_{p\infty}^*=\sup_{t>0} t^{1/p}\mu_t(x).$$

We call it the *non-commutative Lorentz space* associated with $\mathcal{M}$. In particular, we call $Lp\infty(\mathcal{M})$ the *non-commutative weak L^p-space.*

Like the classical Lorentz spaces, each $Lpq\,(\mathcal{M};\tau)$ together with $\|\cdot\|_{pq}^*$ is a quasi-normed linear space, and, whenever $q_1\le q_2$, $Lpq_1\,(\mathcal{M};\tau)$ is continuously included in $Lpq_2\,(\mathcal{M};\tau)$.

By Corollary 1.5, $Lpp\,(\mathcal{M};\tau)$ (together with $\|\cdot\|_{pp}^*$) is exactly the non-commutative L^p-space $L^p(\mathcal{M};\tau)$ ([4]). By Corollary 1.3, $Lpq\,(\mathcal{M};\tau)$ is an $\mathcal{M}$-bimodule in the natural way and it is stable under the adjoint operation (of operators).

We now apply the real interpolation method (the K-method) to the pair $(L^1, L^\infty)=(L^1(\mathcal{M};\tau), L^\infty(\mathcal{M};\tau))$. We recall the K-method of Peetre [1] briefly.

Definition 2.2 (K-functional). For $x\in L^1+L^\infty$, the algebraic sum, and $t>0$, we set

$$K(t, x)\ (=K(t, x\,;L^1, L^\infty)) \\ =\inf\{\|x_1\|+t\,\|x_\infty\|_\infty\,;\ x=x_1+x_\infty,\ x_1\in L^1,\ x_\infty\in L^\infty\}.$$

Here, $\|\cdot\|_\infty$ (resp. $\|\cdot\|_1$) is the uniform norm of $L^\infty(\mathcal{M}\,;\tau)=\mathcal{M}$ (resp. the norm of $L^1(\mathcal{M}\,;\tau)\cong\mathcal{M}_*$).

Definition 2.3 (the K-method of Peetre). For $0<\theta<1$, $1\le q\le\infty$, $(L^1, L^\infty)_{\theta,q}$ is the set of every x in L^1+L^∞ satisfying

$$\|x\|_{\theta,q}=\left[\int_0^t \{t^{-\theta}K(t, x)\}^q\frac{dt}{t}\right]^{1/q}<\infty.$$

This is called the real interpolation space and $(L^1, L^\infty)_{\theta,q}$ together with the norm $\|x\|_{\theta,q}$ is a Banach space.

Here is the main theorem of the section:

Theorem 2.4. *For $1<p<\infty$, $1\le q\le\infty$, the real interpolation space $(L^1, L^\infty)_{\theta,q}$ with $\theta=1-1/p$ is exactly the non-commutative Lorentz space $Lpq(\mathcal{M}\,;\tau)$. Furthermore, $\|\cdot\|_{pq}^*$ and $\|\cdot\|_{\theta,q}$ give rise to the same topology.*

This result is a consequence of the Hardy-Littlewood inequality (see [1] or [8]) and the following result:

Theorem 2.5. *For each $t>0$ and an operator x,*

$$K(t, x)=\int_0^t \mu_s(x)ds.$$

§3. Applications. By Theorem 2.4, the abstract interpolation theorem (the abstract Marcinkiewicz theorem) is available to our non-commutative Lorentz spaces (see [1] for details). We give two applications.

At first, let G be a locally compact *unimodular* group with the left Haar measure dg. We denote the semi-finite von Neumann algebra $\lambda((t))''$ on $L^2(G, dg)$ generated by the left regular representation λ simply by $\mathcal{M}$. The algebra $\mathcal{M}$ admits the canonical trace τ, the dual Haar trace. Following Kunze [4], for a function f on G, we consider the convolution operator $\lambda(f)=f^*$ on $L^2(G)$ as the Fourier transform $\mathcal{F}(f)$. The next result is a generalization of Paley's theorem on Fourier series, which is a slight strengthening of Kunze's version of the Hausdorff-Young theorem [4].

Theorem 3.1. *For $1<p\le 2$, $1/p+1/q=1$, the Fourier transform $\mathcal{F}$ is a bounded operator from $L^p(G\,;dg)$ to $Lpq(\mathcal{M}\,;\tau)$.*

Secondly, we consider Haagerup's L^p-space $L^p(\mathcal{N})$ associated with an *arbitrary* von Neumann algebra $\mathcal{N}$ (not necessarily semi-finite) [3]. We recall that his $L^p(\mathcal{N})$ consists of certain (measurable) operators affiliated with the crossed product $\mathcal{M}=\mathcal{N}\times_\sigma \boldsymbol{R}$ relative to the modular automorphism group and that $\mathcal{M}$ is always a semi-finite von Neumann algebra admitting a relatively invariant trace τ (with respect to the dual action) (see [9]). Although Haagerup's $L^p(\mathcal{N})$ is *not* included in $L^p(\mathcal{M}\,;\tau)$, we have:

Theorem 3.2. *For each* $1<p<\infty$, $L^p(\mathcal{N})$ *is a closed subspace of the non-commutative weak* L^p*-space* $Lp\infty(\mathcal{M};\tau)$. *Furthermore, on* $L^p(\mathcal{N})$, *the following three topologies are all identical*:

(i) *the norm topology of* $L^p(\mathcal{N})$ (introduced in [3]),

(ii) *the topology as a closed subspace of* $Lp\infty(\mathcal{M};\tau)$,

(iii) *the measure topology* (see [5]).

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