70. An Analogue of Paley-Wiener Theorem on a Real Rank 2 Semisimple Lie Group

The Case of 1 Dimensional τ-Sherical Functions

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In the previous paper [4] we obtained an analogue of Paley-Wiener theorem on $SU(2,2)$. In this article we shall give more precise results about this theorem, particularly, replace the condition (C1) in [4] by explicit conditions (cf. (C2)–(C4) in § 5).

1. Notation and assumptions. Let G be a connected semisimple Lie group with finite center and $G=KAN$ an Iwasawa decomposition for G. Let M be the centralizer of A in K and put $P=MAN$. Then P is a minimal parabolic subgroup of G. We denote the Lie algebras by small German letters. Let Σ denote the set of all roots for the pair $(\mathfrak{g}, \mathfrak{a})$ and W the corresponding Weyl group. Let Σ^+ denote the set of all positive roots in Σ and $\mathfrak{a}^+$ the corresponding positive Weyl chamber in $\mathfrak{a}$. Put $\rho=(1/2)\sum_{\beta\in\Sigma^+}\beta$. For simplicity we denote the dual space of $\mathfrak{a}$ by $\mathcal{F}$ and its complexification by $\mathcal{F}^c$. Put $\mathcal{F}^+=\{\lambda\in\mathcal{F}; \langle\lambda,\alpha\rangle>0 \text{ for all } \alpha\in\Sigma^+\}$ and $A^+=\exp\mathfrak{a}^+$.

For any root α in Σ let $\mathfrak{a}_\alpha$ denote the hyperplane of $\alpha=0$ in $\mathfrak{a}$ and put $A_\alpha=\exp\mathfrak{a}_\alpha$. Let L_α denote the centralizer of A_α in G. Then it is easy to see that $L_\alpha=M_\alpha A_\alpha$, where $M_\alpha=\bigcap_{\chi\in X(L_\alpha)}\operatorname{Ker}|\chi|$ ($X(L_\alpha)$ is the group of all continuous homomorphisms of L_α into the multiplicative group of real numbers). Then we can define the parabolic subgroup $P_\alpha=M_\alpha A_\alpha N_\alpha$ such that $N_\alpha\subset N$. Put ${}^*P_\alpha=P\cap M_\alpha$ and ${}^*A_\alpha=A\cap M_\alpha$, ${}^*N_\alpha=N\cap M_\alpha$. Then it is easy to see that ${}^*P_\alpha=M{}^*A_\alpha{}^*N_\alpha$ is a minimal parabolic subgroup of M_α and $\dim{}^*A_\alpha=1$. For this pair $(M_\alpha, {}^*A_\alpha)$ we define ${}^*\rho$, ${}^*\mathcal{F}_\alpha$, ${}^*\mathcal{F}_\alpha^c$, ${}^*\mathcal{F}_\alpha^+$ and ${}^*A_\alpha^+$ by the same way, where $\lambda({}^*a)>0$ for ${}^*a\in{}^*A_\alpha^+$ and $\lambda\in\mathcal{F}^+$.

Let $\tau=(\tau_1,\tau_2)$ be a unitary double representation of K on a finite dimensional Hilbert space V. Let $\mathcal{C}(G,\tau)$ denote the τ-spherical Schwartz space on G and ${}^\circ\mathcal{C}(G,\tau)$ the closed subspace of $\mathcal{C}(G,\tau)$ consisting of all cusp forms. For any parabolic subgroup $Q=M_QA_QN_Q$ of G let $\mathcal{C}_{A_Q}(G,\tau)$ denote the closed subspace of $\mathcal{C}(G,\tau)$ consisting of all wave packets corresponding to Q (cf. [2, § 26]). Let τ_M and $\tau_{M_\alpha}(\alpha\in\Sigma^+)$ denote the restrictions of τ to M and $M_\alpha\cap K$ respectively. Then for the τ_M (resp. τ_{M_α})-spherical Schwartz space $\mathcal{C}(M,\tau_M)$ (resp. $\mathcal{C}(M_\alpha,\tau_{M_\alpha})$) we define the above closed subspaces by the same way.

In this article we accept the following assumptions; (A1) $\dim A=2$, (A2) $\dim V=1$, (A3) $\mathcal{C}(G,\tau)={}^\circ\mathcal{C}(G,\tau)\oplus\mathcal{C}_{A_\alpha}(G,\tau)\oplus\mathcal{C}_A(G,\tau)$ for $\alpha\in CL(\mathcal{F}^+)$, (A4) $\{s\alpha;s\in W\}\cap\Sigma^+\subset\{\alpha,\varepsilon_1,\varepsilon_2\}$, where $\{\varepsilon_1,\varepsilon_2\}$ is the fundamental system of positive roots. In these assumptions, (A2) is essential, for the operators: $\Phi(\nu:a)$ $(a\in A^+)$ and $C(s;\nu)$ $(s\in W)$ (see §§ 3, 4) are $\boldsymbol{C}$-valued meromorphic functions of ν on $\mathcal{F}^c$ and moreover $\Phi(\nu:a)$ is holomorphic on $\mathcal{F}+\sqrt{-1}CL(\mathcal{F}^+)$ under (A2).

2. Fourier transform on $\mathcal{C}(G,\tau)$. Let e_k $(1\le k\le n')$ and $\psi_{\alpha,i}$ $(1\le i\le l')$ (cf. § 3, Theorem 4) denote orthonormal basis for $L_G={}^\circ\mathcal{C}(G,\tau)$ and $L_{M_\alpha}={}^\circ\mathcal{C}(M_\alpha,\tau_{M_\alpha})$ respectively. Then for f in $\mathcal{C}(G,\tau)$ its Fourier transform $E(f)$ is defined as follows;

$$E(f)=((f,e_k))_{k=1}^{n'}\oplus(\hat{f}(\psi_{\alpha,i},\nu_\alpha))_{i=1}^{l'}\oplus\hat{f}(\nu)\qquad(\nu\in\mathcal{F}_\alpha\text{ and }\nu\in\mathcal{F}),$$

where $\hat{f}(\psi_{\alpha,i},\nu_\alpha)=(f,E(P:\psi_{\alpha,i}:\nu_\alpha:\cdot))$ and $\hat{f}(\nu)=(f,E(P:1:\nu:\cdot))$ (see [3]). Obviously, $E(f)$ is contained in $\boldsymbol{C}^{n'}\oplus\mathcal{C}(\mathcal{F}_\alpha)^{l'}\oplus\mathcal{C}(\mathcal{F})$, where $\mathcal{C}(\mathcal{F}_\alpha)$ (resp. $\mathcal{C}(\mathcal{F})$) is the usual Schwartz space on $\mathcal{F}_\alpha$ (resp. $\mathcal{F}$). Here we define the closed subspace $\mathcal{C}(\mathcal{F}_\alpha)_*^{l'}$ of $\mathcal{C}(\mathcal{F}_\alpha)^{l'}$ (resp. $\mathcal{C}(\mathcal{F})_*$ of $\mathcal{C}(\mathcal{F})$) as in [3]. Then we obtained the following theorem in [3].

Theorem 1. *The Fourier transform sets up a homeomorphism of* $\mathcal{C}(G,\tau)={}^\circ\mathcal{C}(G,\tau)\oplus\mathcal{C}_{A_\alpha}(G,\tau)\oplus\mathcal{C}_A(G,\tau)$ *and* $\hat{\mathcal{C}}(G,\tau)=\boldsymbol{C}^{n'}\oplus\mathcal{C}(\mathcal{F}_\alpha)_*^{l'}\oplus\mathcal{C}(\mathcal{F})_*$. *The inverse transform is given by*

$$f(x)=\sum_{k=1}^{n'}(f,e_k)e_k(x)+\sum_{i=1}^{l'}\frac{1}{|W_\alpha|}\int_{\mathcal{F}_\alpha}\mu(\sigma_i,\nu_\alpha)E(P_\alpha:\psi_{\alpha,i}:\nu_\alpha:x)\hat{f}(\psi_{\alpha,i},\nu_\alpha)d\nu_\alpha$$
$$+\frac{1}{|W|}\int_{\mathcal{F}}\mu(\nu)E(P:1:\nu:x)\hat{f}(\nu)d\nu\qquad(x\in G),$$

where W_α is the Weyl group for (G,A_α) and σ_i $(1\le i\le l')$ is the discrete series for M_α such that $\psi_{\alpha,i}$ is the matrix coefficient of σ_i.

Before stating the main result, we have to obtain some information for M_α (see § 3) and calculate residue integrals (see § 4).

3. Analysis on M_α. Let us agree to write the Harish-Chandra expansion of the Eisenstein integral for $(M_\alpha,{}^*P_\alpha)$ as follows;

$$E({}^*P_\alpha:1:{}^*\nu_\alpha:{}^*a)=e^{-{}^*\rho(\log({}^*a))}\{\Phi_\alpha({}^*\nu_\alpha:a)C_\alpha({}^*\nu_\alpha)+\Phi_\alpha(-{}^*\nu_\alpha:{}^*a)C_\alpha(-{}^*\nu_\alpha)\}$$

for ${}^*a\in{}^*A_\alpha^+$ and ${}^*\nu_\alpha\in{}^*\mathcal{F}_\alpha^c$. Let $\{{}^*\xi_i;1\le i\le l\}$ denote the set of all poles of $\Phi_\alpha({}^*\nu_\alpha:{}^*a)C_\alpha({}^*\nu_\alpha)^{*-1}$ on ${}^*\mathcal{F}_\alpha+\sqrt{-1}CL({}^*\mathcal{F}_\alpha^+)$. Then it is easy to see that ${}^*\xi_i\in\sqrt{-1}{}^*\mathcal{F}_\alpha^+$. Here we note that $\mu_\alpha({}^*\nu_\alpha)$, the μ-function for $(M_\alpha,{}^*P_\alpha)$, has at most simple pole or zero at ${}^*\nu_\alpha={}^*\xi_i\neq0$ and $\Phi_\alpha({}^*\nu_\alpha:{}^*a)$ is holomorphic on ${}^*\mathcal{F}_\alpha+\sqrt{-1}CL({}^*\mathcal{F}_\alpha^+)$. Therefore using the relation: $\mu_\alpha({}^*\nu_\alpha)C_\alpha({}^*\nu_\alpha)^*C_\alpha({}^*\nu_\alpha)=1$ $({}^*\nu_\alpha\in{}^*\mathcal{F}_\alpha^c)$, we can obtain the following

Lemma 3. *Each ${}^*\xi_i$ $(1\le i\le l)$ is a simple pole of $\mu_\alpha({}^*\nu_\alpha)$ and $E({}^*P_\alpha:1:{}^*\xi_i:{}^*a)$ $({}^*a\in{}^*A_\alpha^+)$ is of the form:*

(I) $e^{-{}^*\rho(\log({}^*a))}\Phi_\alpha({}^*\xi_i:{}^*a)C_\alpha({}^*\xi_i)\neq0,$ *or*

(II) $e^{-{}^*\rho(\log({}^*a))}\{\Phi_\alpha({}^*\xi_i : {}^*a)C_\alpha({}^*\xi_i)+\Phi_\alpha(-{}^*\nu_\alpha : {}^*a)C_\alpha(-{}^*\nu_\alpha)|_{{}^*\nu_\alpha={}^*\xi_i}\}$.
In both cases $E({}^*P_\alpha : 1 : {}^*\xi_i : m)$ $(m \in M_\alpha)$ *is contained in* ${}^\circ\mathcal{C}(M_\alpha, \tau_{M_\alpha})$.

For simplicity we put $E_i = E({}^*P_\alpha : 1 : {}^*\xi_i : \cdot)$ $(1 \le i \le l)$ and suppose that $E_i \ne 0$ for $1 \le i \le l'$ and zero for otherwise. Then applying the results in the case of real rank 1 (cf. [3]), we conclude the following

Theorem 4. *Each* E_i $(1 \le i \le l')$ *is of the form* (I) *in Lemma* 3 *whose* L^2*-norm is given by* $\|E_i\|^2 = -\mu_{\alpha,i}^{-1}$, *where*

$$\mu_{\alpha,i} = 2\pi\sqrt{-1} \operatorname*{Res}_{{}^*\nu_\alpha={}^*\xi_i} \mu_\alpha({}^*\nu_\alpha)$$

and each E_i $(l'+1 \le i \le l)$ *is of the form* (II). *Moreover*

$$\{\psi_{\alpha,i} = E_i(-\mu_{\alpha,i})^{1/2} ; 1 \le i \le l'\}$$

is an orthonormal basis for ${}^\circ\mathcal{C}(M_\alpha, \tau_{M_\alpha})$.

Let f be in $C_c^\infty(G, \tau)$, where $C_c^\infty(G, \tau)$ is the set of all C^∞, τ-spherical functions on G with compact support. Then it is easy to see from the above results that $\hat{f}(\psi_{\alpha,i}, \nu_\alpha) = (-\mu_{\alpha,i})^{1/2}\hat{f}(\nu_\alpha + {}^*\xi_i)$ $(\nu_\alpha \in \mathcal{F}_\alpha,\ 1 \le i \le l')$. According to Theorem 1, we decompose f as $f = f_0 + f_1 + f_2$, where $f_0 \in {}^\circ\mathcal{C}(G, \tau)$, $f_1 \in \mathcal{C}_{A_\alpha}(G, \tau)$ and $f_2 \in \mathcal{C}_A(G, \tau)$. Then using the inverse transform for f_1 and Lemma 5 in [4], we have the following

Corollary 5.

$$f_1(x) = \frac{-1}{|W_\alpha|} \sum_{i=1}^{l'} \int_{\mathcal{F}_\alpha} 2\pi\sqrt{-1} \operatorname*{Res}_{{}^*\nu_\alpha={}^*\xi_i} \mu_\alpha(\nu_\alpha + {}^*\nu_\alpha) \times E(P : 1 : \nu_\alpha + {}^*\xi_i : x)\hat{f}(\nu_\alpha + {}^*\xi_i)d\nu_\alpha.$$

4. Residue integrals. Let f be in $C_c^\infty(G, \tau)$ and put $f = f_0 + f_1 + f_2$ as before. Then by Theorem 1,

$$f_2(x) = \frac{1}{|W|} \int_{\mathcal{F}} \mu(\nu)E(P : 1 : \nu : x)\hat{f}(\nu)d\nu \qquad (x \in G).$$

Here we replace $E(P : 1 : \nu : a)$ $(a \in A^+)$ by its Harish-Chandra expansion:

$$E(P : 1 : \nu : a) = e^{-\rho(\log(a))} \sum_{s \in W} \Phi(s\nu : a)C(s ; \nu) \qquad (\nu \in \mathcal{F}^c)$$

and shift the integral line from $\mathcal{F}$ to $\mathcal{F} + \sqrt{-1}\delta$, where $\delta \in \mathcal{F}^+$ such that $\mu(\nu)\Phi(\nu : a)C(1 ; \nu)$ is holomorphic on $\mathcal{F} + \sqrt{-1}(\delta + \mathcal{F}^+)$. Then using the same arguments in [4], we can obtain that

$$\begin{aligned} f_2(a) = &\int_{\mathcal{F}+\sqrt{-1}\delta} e^{-\rho(\log(a))}\Phi(\nu : a)C(1 ; \nu)^{*-1}\hat{f}(\nu)d\nu \\ &+ \sum_{i=1}^{l'} \sum_{s \in W_0} \int_{\mathcal{F}} e^{-\rho(\log(a))} 2\pi\sqrt{-1} \operatorname*{Res}_{{}^*\nu_\alpha={}^*\xi_i} \mu(\nu_\alpha + {}^*\nu_\alpha) \\ &\qquad \times \Phi(s(\nu_\alpha + {}^*\xi_i) : a)C(1 ; s(\nu_\alpha + {}^*\xi_i))\hat{f}(s(\nu_\alpha + {}^*\xi_i))d\nu_\alpha \\ &- \sum_{i=1}^{l'} \sum_{j=1}^{l''} 4\pi^2 \operatorname*{Res}_{\nu_\alpha={}^*\xi_i} \operatorname*{Res}_{{}^*\nu_\alpha=\zeta_{ij}} e^{-\rho(\log(a))}\Phi(\nu_\alpha + {}^*\nu_\alpha)C(1 ; \nu_\alpha + {}^*\nu_\alpha)^{*-1}\hat{f}(\nu_\alpha + {}^*\nu_\alpha), \end{aligned}$$

where $W_0 = \{s \in W ; s\alpha \in \Sigma^+\}$ and $\{({}^*\xi_i, \zeta_{ij}) ; 1 \le i \le l', 1 \le j \le l''\}$ is the set of intersections of singular hyperplanes of $\Phi(\nu : a)C(1 ; \nu)^{*-1}$ which arise when we shift the integral line. For simplicity we denote the first two integrals by f_2^c and $I_{f,i}^s$ respectively and the last term by RR_f. Then

$$f_2 = f_2^c + \sum_{i=1}^{l'} \sum_{s \in W_0} I_{f,i}^s - RR_f.$$

Obviously, using the same arguments in the classical Paley-Wiener theorem on an Euclidean space, we see that f_2^c belongs to $C_c^\infty(G,\tau)$.

For each i, j $(1<j\leq l'', 1\leq i\leq l')$ let us suppose that $\Phi(\nu:a)C(1;\nu)^{*-1}$ has a pole of m_{ij}-th order at $\nu=\nu_\alpha+\zeta_{ij}$ $(\nu_\alpha\in\mathcal{F}_\alpha)$. Here we put

$$S=\{D^{(0,m)}(^*\xi_i,\zeta_{ij})E(P:1:\nu_\alpha+{}^*\nu_\alpha:\cdot)\,;\,0\leq m\leq m_{ij}-1,\quad 1\leq i\leq l',\,1\leq j\leq l''\},$$

where

$$D^{(n,m)}(\xi,\zeta)=\frac{\partial^n}{\partial\nu_\alpha^n}\bigg|_{\nu_\alpha=\xi}\frac{\partial^m}{\partial^*\nu_\alpha^m}\bigg|_{^*\nu_\alpha=\zeta}.$$

Let $\{E_{(p)}=D^{(0,m_{(p)})}(^*\xi_{(p)},\zeta_{(p)})E(P:1:\cdot:x)\,;\,1\leq p\leq\gamma\}$ be a maximal linearly independent subset of S. Then each element of S can be written as a linear combination of $E_{(p)}$ $(1\leq p\leq\gamma)$, that is,

$$D^{(0,m)}(^*\xi_i,\zeta_{ij})E(P:1:\cdot:x)=\sum_{p=1}^{\gamma}A_{m,i,j,p}E_{(p)}(x)\qquad(x\in G,\ A_{m,i,j,p}\in\boldsymbol{C}).$$

As in [3], we choose $h_{(p)}$ $(1\leq p\leq\gamma)\in C_c^\infty(G,\tau)$ such that $(h_{(p)},E_{(q)})=\delta_{pq}$ $(1\leq p,q\leq\gamma)$ and put $A_{p,k}=(h_p,e_k)$ $(1\leq p\leq\gamma, 1\leq k\leq n')$.

Lemma 6. *For f in $C_c^\infty(G,\tau)$ we put $F=f-\sum_{p=1}^{\gamma}(f,E_{(p)})h_{(p)}$. Then $RR_F=0$.*

This lemma is proved by the same way in [3, I, cf. (4.16)].

In what follows we shall prove that $F_0=0$. Here we note that each $E(P_\alpha:\psi_{\alpha,i}:\nu_\alpha:x)=(-\mu_{\alpha,i})^{1/2}E(P:1:\nu_\alpha+{}^*\xi_i:x)$ $(1\leq i\leq l')$ satisfies the weak inequality on G for $\nu_\alpha\in\mathcal{F}_\alpha$. Then using the Harish-Chandra expansion of the right hand side, we can deduce the first relation in the following proposition and using the definition of $\Phi(\nu:a)$ directly the second one.

Proposition 7. (i)

$$\sum_{i=1}^{l'}\sum_{s\in W_0}I_{F,i}^s=\frac{1}{|W_\alpha|}\sum_{i=1}^{l'}\int_{\mathcal{F}_\alpha}2\pi\sqrt{-1}\operatorname*{Res}_{^*\nu_\alpha={}^*\xi_i}\mu(\nu_\alpha+{}^*\nu_\alpha)\times E(P:1:\nu_\alpha+{}^*\xi_i:\cdot)\hat{F}(\nu_\alpha+{}^*\xi_i)d\nu_\alpha-\sum_{i=1}^{l'}\sum_{s\in W_0'}J_{F,i}^s,$$

where $W_0'=W-W_0$ and for $s\in W_0'$,

$$J_{F,i}^s(a)=\int_{\mathcal{F}_\alpha}2\pi\sqrt{-1}\Big\{\operatorname*{Res}_{^*\nu_\alpha={}^*\xi_i}\mu(\nu_\alpha+{}^*\nu_\alpha)\Big\}\Phi(s(\nu_\alpha+{}^*\nu_\alpha):a)\times C(s;\nu_\alpha+{}^*\nu_\alpha)|_{^*\nu_\alpha={}^*\xi_i}\hat{F}(\nu_\alpha+{}^*\xi_i)d\nu_\alpha\qquad(a\in A^+).$$

(ii) *For each $s\in W_0'$ and $1\leq i\leq l'$ there exists a meromorphic function P_i^s on $\mathcal{F}_\alpha$ such that*

$$J_{F,i}^s(a)=\exp\{-(\rho+\sqrt{-1}s^*\xi_i)(\log(a))\}H_{F,i}^s(a)\qquad(a\in A^+),$$

where

$$H_{F,i}^s(a)=\frac{1}{|W_\alpha|}\int_{\mathcal{F}_\alpha}2\pi\sqrt{-1}\Big\{\operatorname*{Res}_{^*\nu_\alpha={}^*\xi_i}\mu(\nu_\alpha+{}^*\nu_\alpha)\Big\}P_i^s(\nu_\alpha)\hat{F}(\nu_\alpha+{}^*\xi_i)\times\exp\{\sqrt{-1}s\nu_\alpha(\log(a))\}d\nu_\alpha.$$

Then by Corollary 5 and the above relation (i), we see that

$$(*)\qquad \sum_{i=1}^{l'}\sum_{s\in W_0} I_{F,i}^s = -F_1 - \sum_{i=1}^{l'}\sum_{s\in W_0'} J_{F,i}^s$$

and thus,

$$F = F_0 + F_2^c - \sum_{i=1}^{l'}\sum_{s\in W_0} J_{F,i}^s.$$

Moreover we obtain the following lemma which was used in [4] without a proof.

Lemma 8. *$J_{F,i}^s = 0$ for all $s \in W_0'$ and $1 \le i \le l'$.*

To prove this lemma it is enough to prove that $\Phi(s(\nu_\alpha + {}^*\nu_\alpha) : a)$ ($a \in A^+$, $\nu_\alpha \in \mathcal{F}_\alpha$) is holomorphic at ${}^*\nu_\alpha = {}^*\xi_i$ for all $s \in W_0'$ and $1 \le i \le l'$. This fact is obtained from the definitions of $\Phi(\nu : a)$ and $\Phi_\alpha({}^*\nu_\alpha : {}^*a)$ for $s \in W_0'$ such that $s\alpha = -\varepsilon_1$ or $-\varepsilon_2$, and from the relation $(*)$, Proposition 7 (ii) and the following proposition for $s \in W_0'$ such that $s\alpha = -\alpha$.

Proposition 9. *There exists a function g in $C_c^\infty(G, \tau)$ such that (1) $RR_g = 0$, (2) $g_0 = 0$, (3) $\hat{g}(\nu_\alpha + {}^*\xi_i) \ne 0$ for all $1 \le i \le l'$, (4) when $P_i^s \ne 0$ for $s \in W_0'$ and $1 \le i \le l'$, then $J_{g,i}^s \ne 0$.*

Therefore using Lemma 8 and the relation $(*)$, we see that $F = F_0 + F_2^c$. Since $F - F_2^c$ has compact support and F_0 is real analytic on G, it is easy to see that $F = F_2^c$ and $F_0 = 0$. In particular, since f is an arbitrary function in $C_c^\infty(G, \tau)$, the following proposition is obtained.

Proposition 10.

$$e_k = \sum_{p=1}^{r} A_{p,k} E_{(p)} \qquad (1 \le k \le n').$$

5. An analogue of Paley-Wiener theorem. We shall define the subspace of $\hat{C}(G, \tau)$ which will be the set of Fourier transforms of $C_c^\infty(G, \tau)$. Let $\mathcal{H}$ be the set of all $b = (b_k)_{k=1}^{n'} \oplus (\beta_i(\nu_\alpha))_{i=1}^{l'} \oplus \beta(\nu)$ in $\hat{C}(G, \tau)$ satisfying the following conditions;

(C1) each β_i ($1 \le i \le l'$) and β can be extended to entire holomorphic functions, which are exponential type, on $\mathcal{F}_\alpha^c$ and $\mathcal{F}^c$ respectively,

(C2) $\beta_i(\nu_\alpha) = \beta(\nu_\alpha + {}^*\xi_i) \qquad (1 \le i \le l',\ \nu_\alpha \in \mathcal{F}_\alpha^c)$,

(C3) $b_k = \sum_{p=1}^{r} A_{p,k} D^{(0,m)}({}^*\xi_{(p)}, \zeta_{(p)})\beta \qquad (1 \le k \le n')$,

(C4) $D^{(0,m)}({}^*\xi_i, \zeta_{ij})\beta = \sum_{p=1}^{r} A_{m,i,j,p} D^{(0,m(p))}({}^*\xi_{(p)}, \zeta_{(p)})\beta$
$(1 \le i \le l',\ 1 \le j \le l'')$.

Then our main result can be stated as follows.

Theorem 2. *The Fourier transform sets up a bijection between $C_c^\infty(G, \tau)$ and $\mathcal{H}$.*

(*Pr*) As has been seen in § 4, it is easy to see that the mapping: $f \mapsto E(f)$ is injective one of $C_c^\infty(G, \tau)$ to $\mathcal{H}$. Therefore it remains to prove the surjectivity. Let $b = (b_k)_{k=1}^{n'} \oplus (\beta_i)_{i=1}^{l'} \oplus \beta$ be in $\mathcal{H}$ and put $f = E^{-1}(b)$. Our purpose is to prove that f is contained in $C_c^\infty(G, \tau)$. Here we define F by $F = f - \sum_{p=1}^{r} D^{(0,m(p))}({}^*\xi_{(p)}, \zeta_{(p)})\beta h_{(p)}$. Obviously, $E(F)$ belongs to $\mathcal{H}$ and thus satisfies the above conditions: (C1)–(C4).

Then, applying the same arguments as before, we see that $F=F_2^c$, particularly, F has compact support. Therefore f has also compact support on G. Q.E.D.

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