68. *The Incompressible Limit of Compressible Fluid Motions in a Bounded Domain*

By Rentaro AGEMI
Department of Mathematics, Hokkaido University

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1. Introduction. The aim of this note is to show the convergence of inviscid compressible fluids in a bounded domain to their incompressible limit as the Mach number becomes small. For the periodic fluid motions see Ebin [2] and Klainerman-Majda [4].

2. Statements of results. We consider the equations of inviscid compressible fluid motions involving the Mach number as parameter in a bounded domain Ω of R^3 with smooth boundary $\partial\Omega$;

$$(1)_\lambda \quad \begin{array}{ll} \partial_t v^\lambda + (v^\lambda \nabla) v^\lambda + (\lambda^2 \nabla p(\rho^\lambda))/\rho^\lambda = 0 & \\ \partial_t \rho^\lambda + \operatorname{div}(\rho^\lambda v^\lambda) = 0 & \text{in } (0,T)\times\Omega, \\ (v^\lambda(0),\ \rho^\lambda(0)) = (v_0, \rho_0) & \text{on } \Omega, \\ \langle v^\lambda, n\rangle = 0 & \text{on } (0,T)\times\partial\Omega. \end{array}$$

Here λ is the reciprocal of the Mach number and $\langle v, n\rangle$ is the inner product of velocity field v and the unit outernormal n to Ω. Moreover we assume that the fluid motion is isentropic, i.e., the pressure p is a smooth function of the density ρ only and its derivative p' in ρ is positive.

We shall show that the limit v^∞ of v^λ as $\lambda \to \infty$ satisfies the equations of homogeneous incompressible fluid motion;

$$(2) \quad \begin{array}{ll} P(\partial_t v^\infty + (v^\infty \nabla) v^\infty) = 0 & \\ \operatorname{div} v^\infty = 0 & \text{in } (0,T)\times\Omega, \\ v^\infty(0) = v_0 & \text{on } \Omega, \\ \langle v^\infty, n\rangle = 0 & \text{on } (0,T)\times\partial\Omega, \end{array}$$

where P is the orthogonal projection on solenoidal vector fields.

When we discuss the incompressible limit, we may assume that

(3) v_0 is solenoidal and ρ_0 is constant.

By definition of P there exists a pressure function p^∞ such that

$$\partial_t v^\infty + (v^\infty \nabla) v^\infty + \nabla p^\infty / \rho_0 = 0.$$

If $(v^\lambda, \rho^\lambda)$ converges to (v^∞, ρ^∞) as $\lambda \to \infty$, then $\lambda^2 \nabla p(\rho^\lambda)/\rho^\lambda$ converges to $\nabla p^\infty/\rho_0$ and ∇p^∞ vanishes at $t=0$. Thus we can also assume that

(4) $\partial_t v^\lambda(0) = -(v_0 \nabla) v_0$ is solenoidal.

Furthermore we assume the compatibility conditions up to order 3 for the initial boundary value problem $(1)_\lambda$;

$(5)_k$ $\langle \partial_t^k v^\lambda(0), n\rangle = 0 \quad (k=0,1,2)$ on $\partial\Omega$.

We note that the assumptions (3) and (4) imply $(5)_0$ and $(5)_1$.

The main result is

Theorem. *Suppose that v_0 belongs to $H^3(\Omega, R^3)$ and (v_0, ρ_0) satisfy the conditions* (3), (4) *and* $(5)_2$. *Then there exists a positive constant T_1 independent of λ such that the initial boundary value problem* $(1)_\lambda$ *has a unique solution $(v^\lambda, \rho^\lambda)$ with $\partial_t^k(v^\lambda, \rho^\lambda) \in L^\infty([0, T_1], H^{3-k}(\Omega, R^4))$ $(k=0,1,2,3)$ and $(v^\lambda, \rho^\lambda)$ converges in the following sense to the solution (v^∞, ρ_0) of* (2) *with $\partial_t^k v^\infty \in L^\infty([0, T_1], H^{3-k}(\Omega, R^3))$ $(k=0,1,2,3)$ as $\lambda \to \infty$;*

$$\partial_t^k(v^\lambda, \rho^\lambda) \to \partial_t^k(v^\infty, \rho_0) \quad \textit{weak-star in } L^\infty([0, T_1], H^{3-k}(\Omega, R^4)) \qquad (k=0,1,2).$$

Remark. We can also show the similar results for the fluid motions involving the equation of entropy S;

$$\partial_t S^\lambda + (v^\lambda \nabla) S^\lambda = 0, \qquad \rho^\lambda = \rho(p^\lambda, S^\lambda),$$
$$\frac{\partial \rho}{\partial p} > 0, \qquad \frac{\partial \rho}{\partial S} \neq 0.$$

Here we consider the pressure p as unknown and its initial value is constant. In this case the limit (v^∞, ρ^∞) satisfies the equations of (inhomogeneous) incompressible fluid motion;

$$(6)\quad \begin{cases} P(\rho^\infty(\partial_t v^\infty + (v^\infty \nabla) v^\infty)) = 0 & \\ \operatorname{div} v^\infty = 0 & \text{in } (0, T)\times\Omega, \\ \partial_t \rho^\infty + (v^\infty \nabla)\rho^\infty = 0 & \\ (v^\infty(0), \rho^\infty(0)) = (v_0, \rho_0) & \text{on } \Omega, \\ \langle v^\infty, n\rangle = 0 & \text{on } (0, T)\times\partial\Omega, \end{cases}$$

since the equation $\partial_t S^\infty + (v^\infty \nabla) S^\infty = 0$ is equivalent to $\partial_t \rho^\infty + (v^\infty \nabla)\rho^\infty = 0$.

The details will be published elsewhere.

3. Outline of proofs. Theorem can be proved by using the methods in [1] and [3]. In particular, the energy integral [1, § 6] plays an important role. Introduce new functions $g^\lambda = \log \rho^\lambda$ and $a(g^\lambda) = p'$ $(\exp(g^\lambda))$, we obtain a system of equations equivalent to $(1)_\lambda$ for unknowns (v^λ, g^λ);

$$(7)_\lambda\quad \begin{cases} \partial_t v^\lambda + (v^\lambda \nabla) v^\lambda + \lambda^2 a(g^\lambda) \nabla g^\lambda = 0 & \text{in } (0, T)\times\Omega, \\ \partial_t g^\lambda + (v^\lambda \nabla) g^\lambda + \operatorname{div} v^\lambda = 0 & \\ (v^\lambda(0), g^\lambda(0)) = (v_0, g_0) & \text{on } \Omega, \\ \langle v^\lambda, n\rangle = 0 & \text{on } (0, T)\times\partial\Omega. \end{cases}$$

Since the boundary is characteristic for $(7)_\lambda$, we transform $(7)_\lambda$ to an equivalent system of integro-differential equations for $(w^\lambda, \nabla f^\lambda, g^\lambda)$ with $v^\lambda = w^\lambda + \nabla f^\lambda$ and $w^\lambda = P v^\lambda$;

$$(8)_\lambda\quad \begin{cases} (\partial_t + v^\lambda \nabla)^2 g^\lambda - \lambda^2 \operatorname{div}(a(g^\lambda)\nabla g^\lambda) = \operatorname{tr}((Dv^\lambda)^2) & \\ \Delta f^\lambda = -(\partial_t + v^\lambda \nabla) g^\lambda + \dfrac{1}{|\Omega|}\displaystyle\int_\Omega (\partial_t + v^\lambda \nabla) g^\lambda dx & \text{in } (0, T)\times\Omega, \\ \partial_t w^\lambda + P((v^\lambda \nabla) w^\lambda + (w^\lambda \nabla)\nabla f^\lambda) = 0 & \\ g^\lambda(0) = g_0, \quad \partial_t g^\lambda(0) = 0, \quad w^\lambda(0) = P v_0 & \text{on } \Omega, \\ \langle \lambda^2 a(g^\lambda)\nabla g^\lambda + (v^\lambda \nabla) v^\lambda, n\rangle = 0 & \text{on } (0, T)\times\partial\Omega, \end{cases}$$

where $|\Omega|$ is the volume of Ω and $Dv=(\partial v_j/\partial x_k\,;\, j, k=1,2,3)$.

A key of proofs is to show the uniform estimates for the solution (v^λ, g^λ) of $(8)_\lambda$:

There exist positive constants T_1, C and λ_1 independent of λ such that for any t $(0\leqq t\leqq T_1)$ and any $\lambda\geqq\lambda_1$

(9) $\lambda(\|\nabla g^\lambda(t)\|_2+\|\partial_t\nabla g^\lambda(t)\|_1+\|\partial_t^2\nabla g^\lambda(t)\|_0\leqq C,$

(10) $\|v^\lambda(t)\|_3+\|\partial_t v^\lambda(t)\|_2+\|\partial_t^2 v^\lambda(t)\|_1\leqq C,$

(11) $\|g^\lambda(t)-g_0\|_3+\|\partial_t g^\lambda(t)\|_2+\|\partial_t^2 g^\lambda(t)\|_1\leqq C.$

In order to replace C in (11) by C_1/λ we use the conservation law of mass which follows from the second equation of $(1)_\lambda$

$$\int_\Omega \partial_t\,(\exp\,(g^\lambda))dx=0$$

and Poincaré lemma

$$\|h\|_0^2\leqq c_\Omega\left(\|\nabla h\|_0^2+\left(\int_\Omega h(x)\,dx\right)^2\right).$$

Set $h=\partial_t^k\,(\exp\,(g^\lambda))$ $(k=1,2)$. Then it follows from (9) and (10) that

(12) $\lambda(\|g^\lambda(t)-g_0\|_3+\|\partial_t g^\lambda(t)\|_2+\|\partial_t^2 g^\lambda(t)\|_1\leqq C_1.$

Since $a(g^\lambda)\nabla g^\lambda$ is gradient, Theorem follows from the uniform stability (9), (10) and (12).

References

[1] R. Agemi: The initial boundary value problem for inviscid barotropic fluid motion. Hokkaido Math. J., **10**, 156–182 (1981).

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[4] S. Klainerman and A. Majda: Singular limits of quasi-linear hyperbolic system with large parameters and the incompressible limit of compressible fluids (to appear in Comm. Pure Appl. Math.).