6. *The Lax-Milgram Theorem for Banach Spaces. II*

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§0. This paper is a sequel to [1] wherein we proved the Lax-Milgram theorem for a continuous and coercive bilinear form on a Banach space over $\boldsymbol{R}$. Here, we deal with the question of how far coercivity is necessary for the validity of the Lax-Milgram theorem. We construct a counter-example to show that coercivity is not necessary even on Hilbert spaces for the validity of the Lax-Milgram theorem. However, we prove that it is necessary in case the bilinear form 'a' is symmetric and positive-definite in the sense that $a(x, x)>0\ \forall x \neq 0$.

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§1. Let V be a normal space over $\boldsymbol{R}$. Let $\|x\|$ denote the norm of the element $x \in V$. Let V' be the dual of V. Let 'a' be a continuous bilinear form on V. We do not necessarily assume that $\mathrm{a}(x, x)>0\ \forall x \neq 0$.

We have the maps A and B from V to V' defined as $Ax(y)=a(y, x)$ and $Bx(y)=a(x, y)$. A and B are both continuous from V to V'. Let A^*(resp. B^*) be the adjoint of A(resp. B). A^* and B^* are maps from V'' (the double dual of V) to V'. It is easily seen that B (resp. A) is the restriction of A^*(resp. B^*) to V.

Motivated by the Lax-Milgram [1], we make the following definition.

Definition 1. V is said to have the *right* (resp. *left*) *Lax-Milgram property* with respect to 'a' if $\forall f \in V'$, $\exists$ a unique $u \in V$ such that $f(v)=a(v, u)$ (resp. $f(v)=a(u, v)$) $\forall v \in V$.

When 'a' is symmetric, it is clear that to say V has the right Lax-Milgram property is the same as to say that V has the left Lax-Milgram property. In this case, we simply speak of "*the Lax-Milgram property*".

The definition states that V has the right (resp. left) Lax-Milgram property with respect to 'a' iff A(resp. B) is one-one and onto, i.e. iff A(resp. B) is an isomorphism of V and V' in the algebraic sense.

Definition 2. A bilinear form 'a' on V is said to be *non-degenerate* if $\forall y \neq 0$, $\exists x, z \in V$ such that $a(x, y) \neq 0$ and $a(y, z) \neq 0$.

If 'a' has the property that $a(x, x)>0\ \forall x \neq 0$, then it is clearly non-degenerate.

Theorem 1. *Let V be a Banach space. Then, for V to have right (resp. left) Lax-Milgram property with respect to 'a', it is necessary that $\exists\ c>0$ such that $\forall x \in V$,*

$$(*): \quad \sup_{\substack{y\in V\\ y\neq 0}} \frac{|a(y,x)|}{\|y\|} \geqslant c\|x\|.$$

$$\left(\text{resp. } (*)': \sup_{\substack{y\in V\\ y\neq 0}} \frac{|a(x,y)|}{\|y\|} \geqslant c\|x\|.\right)$$

Further, if V is reflexive and 'a' is non-degenerate, then $(*)$ ($(*)'$) is also sufficient for V to have right (resp. left) Lax-Milgram property with respect to 'a'.

Proof. We shall prove the theorem for the case of the right Lax-Milgram property. The proof for the other case is analogous.

Suppose V has the right Lax-Milgram property with respect to 'a'. Then, the map A from V to V' is an algebraic isomorphism. Since A is continuous, by the Banach open mapping theorem, A is also a topological isomorphism. That is, the inverse A^{-1} from V' to V is continuous. This implies $(*)$.

Let now $(*)$ hold and let V be reflexive. Then, $c\|x\|\leqslant\|Ax\| \leqslant\|A\|\,\|x\|\forall x \in V$. Thus, we have inequalities analogous to (I) of §1 in [1]. From these inequalities, it follows that A is one-one and that $A(V)$ is a closed subspace of V'. Now, exactly as in the proof of the sufficiency part of Theorem 1 in [1], it can be proved, using the reflexivity of V and the fact that 'a' is non-degenerate, that A is onto.

Q.E.D.

Proposition 1. *Let V be a reflexive Banach space. Let V have the right (resp. left) Lax-Milgram property with respect to 'a'. Then, V has also the left (resp. right) Lax-Milgram property with respect to 'a'.*

Proof. We shall prove the theorem for the right Lax-Milgram property. The proof for the other case is analogous.

Since V and V' are Banach spaces and since V has the right Lax-Milgram property, it follows by the open mapping theorem that A is a topological isomorphism. Therefore, $A^*: V''\to V'$ is also an isomorphism. But, since V is reflexive, $V''=V$ and since B is A^* restricted to V, it follows that B from V to V' is also an isomorphism, implying that V has the left Lax-Milgram property. Q.E.D.

Proposition 2. *Let V be a Banach space. Suppose V has right and left Lax-Milgram properties with respect to 'a'. Then V is reflexive.*

Proof. We are given that both A and B are topological isomorphisms from V to V'. Since A is a topological isomorphism it follows that A^* from V'' to V' is also an isomorphism. But A^* restricted to V is B and B is an isomorphism of V and V'. Hence A^* is an isomorphism of V'' and V' as well as of V and V'. From this it follows that $V''=V$ implying that V is reflexive. Q.E.D.

Corollary. *If 'a' is symmetric and V has the Lax-Milgram property with respect to 'a', then V is reflexive.*

We recall from [1] that 'b' is the symmetric bilinear form defined as

$$b(x, y)=\frac{a(x, y)+a(y, x)}{2}.$$

Further, if $a(x, x)>0\ \forall x\neq 0$, then V_b is the pre-Hilbert space whose underlying vector space is V and inner-product is 'b'.

Theorem 2. *Let V be a Banach space. Let 'a' be a continuous bilinear form on V such that* $a(x, x)>0\ \forall x\neq 0$. *Then, the following are equivalent.*

(i) *'a' is coercive.*

(ii) *'a' is continuous on* $V_b\times V_b$ *and V has the right Lax-Milgram property with respect to 'a'.*

(iii) *'a' is continuous on* $V_b\times V_b$ *and V has the left Lax-Milgram property with respect to 'a'.*

Proof. It is easy to see that continuity and coercivity of 'a' imply that 'a' is continuous on $V_b\times V_b$. It is proved in the corollary to Theorem 1 in [1] that if 'a' is continuous and coercive, then V has both right and left Lax-Milgram properties with respect to 'a'. Hence, (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii).

We shall prove that (ii)$\Rightarrow$(i).

Since 'a' is continuous, $\exists M_1<+\infty$ such that $|a(x, y)|\leqslant M_1\|x\|\,\|y\|\,\forall x, y\in V$.

Since 'a' is continuous on $V_b\times V_b$, $\exists M_2<+\infty$ such that $|a(x, y)|\leqslant M_2\sqrt{a(x, x)}\sqrt{a(y, y)}$.

Since V has the right Lax-Milgram property with respect to 'a', by Theorem 1, $\exists c>0$ such that

$$\forall x\in V,\ \sup_{\substack{y\in V\\ y\neq 0}}\frac{|a(y, x)|}{\|y\|}\geqslant c\|x\|.$$

Therefore,

$$\begin{aligned}|a(y, x)|&\leqslant M_2\sqrt{a(x, x)}\sqrt{a(y, y)}\\ &\leqslant M_2\sqrt{M_1}\sqrt{a(x, x)}\,\|y\|\end{aligned}$$

$$\Rightarrow\sup_{\substack{y\in V\\ y\neq 0}}\frac{|a(y, x)|}{\|y\|}\leqslant M_2\sqrt{M_1}\sqrt{a(x, x)}.$$

Hence, $c\|x\|\leqslant M_2\sqrt{M_1}\sqrt{a(x, x)}\,\forall x\in V$. This means that '$a$' is coercive. Hence (ii)$\Rightarrow$(i). Similarly (iii)$\Rightarrow$(i). Hence, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii). Q.E.D.

Corollary. *Let 'a' be a symmetric bilinear form on the Banach* *space V such that $a(x, x)>0$ $\forall x \neq 0$. Then, V has the Lax-Milgram property with respect to 'a' iff 'a' is coercive.*

Proof. Sufficient to prove only that 'a' is coercive if V has the Lax-Milgram property with respect to 'a'. Since 'a' is symmetric and $a(x, x)>0$ $\forall x \neq 0$, it follows that

$\forall x, y \in V, |a(x, y)| \leqslant \sqrt{a(x, x)}\sqrt{a(y, y)}$. (Schwarz' inequality.)

Therefore, 'a' is continuous on $V_b \times V_b$. Hence, the conclusion follows immediately from the above theorem. Q.E.D.

Proposition 3. *Let V be a Banach space over $\boldsymbol{R}$ and let 'a' be a continuous bilinear form on V such that $a(x, x)>0$ $\forall x \neq 0$. Then, 'a' is coercive iff V has the Lax-Milgram property with respect to 'b'.*

Proof. Suppose V has the Lax-Milgram property with respect to 'b'. Since 'b' is symmetric and $\mathrm{b}(x, x)=\mathrm{a}(x, x)>0$ $\forall x \neq 0$ it follows from the corollary to Theorem 2 that 'b' is coercive. This means 'a' is coercive.

Conversely, suppose 'a' is coercive. Then, V and V_b are isomorphic. Since V is a Banach space, it follows that V_b is complete. i.e. V_b is a Hilbert space. Therefore, the Riesz representation theorem implies that V_b has the Lax-Milgram property with respect to 'b'. Since V and V_b have the same dual, it follows that V also has the Lax-Milgram property with respect to 'b'. Q.E.D.

The following is a natural question.

Question. Let 'a' be a continuous bilinear form on a Banach space V such that $a(x, x)>0$ $\forall x \neq 0$. Suppose V has both right and left Lax-Milgram properties with respect to 'a', then is 'a' coercive?

We answer this question in the negative even in the case of Hilbert spaces, by means of an example.

We shall construct an isomorphism S of H and H where H is the Hilbert space $l^2 \oplus l^2$, such that $\langle Sx, x\rangle>0$ $\forall x \neq 0$ and such that $S+S^*$ is not an isomorphism. (Here, $\langle\, ,\rangle$ is the inner product in H.) Then, the bilinear form 'a' given by $a(x, y)=\langle Sx, y\rangle$ provides a counter-example to the above question. For, by Proposition 3, 'a' coercive will imply that H has the Lax-Milgram property with respect to 'b'. It is easily seen that H has the Lax-Milgram property with respect to 'b' iff $S+S^*$ is an isomorphism. (More generally, a Banach space V has the Lax-Milgram property with respect to 'b' iff $A+B$ is an isomorphism of V and V'.)

Construction of the counter-example. Let $H=l^2 \oplus l^2$. The inner-product $\langle x, y\rangle$ of $x=x_1+x_2$ and $y=y_1+y_2$ is given by $\langle\!\langle x_1, y_1\rangle\!\rangle+\langle\!\langle x_2, y_2\rangle\!\rangle$ where $\langle\!\langle\, ,\rangle\!\rangle$ is the inner-product in l^2. Let A be the linear map taking an element of the form $u+v$ to $-v+u$ where $u, v \in l^2$. It is easily seen that A is an isometric isomorphism of H and that $\langle Ax, x\rangle=0$ $\forall x \in H$.

It follows that $\|A\|=1=\|A^{-1}\|$ and $A^*=-A$.

Let $(\lambda_n)_{n\in N}$ be a sequence of positive real numbers tending to zero such that $\sup_{n\in N}\lambda_n<1$. Let $(e_n)_{n\in N}$ be an orthonormal basis of H. Let T be the operator from H to H taking $x=\sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ to $\sum_{n=1}^{\infty}\lambda_n\langle x, e_n\rangle e_n$. Then, it is easily seen that $\|T\|<1$, $T=T^*$ and $\langle Tx, x\rangle>0$ $\forall x\neq 0$. Since $\forall n\in N$, λ_n is an eigenvalue and $\lambda_n\to 0$, it follows that T is not invertible.

Let $S=A+T$. Then, since A is invertible, $\|A^{-1}\|=1$ and $\|T\|<1$, S is invertible. $S^*=A^*+T^*=A^*+T$. $S+S^*=A+A^*+2T=2T$ as $A+A^*=0$. Hence, $S+S^*$ is not invertible as T is not.

This finishes the construction of the counter-example.

Let V be a Banach space. Let 'a' be a continuous bilinear form on V. The above counter-example shows that in general, $A+B$ is not an isomorphism even if A and B are. However, we have the following

Proposition 4. *Let V be a Banach space. Let 'a' be a continuous bilinear form. Let V have both right and left Lax-Milgram properties with respect to 'a'. Let further the bilinear form 'b' be non-degenerate. Then, $A+B$ is one-one and $(A+B)(V)$ is dense in V'.*

Proof. It is easily seen that $A+B$ is one-one because 'b' is non-degenerate.

If $(A+B)(V)$ is not dense, $\exists\beta\in V''$, $\beta\neq 0$ such that β vanishes on $(A+B)(V)$. Since V has both left and right Lax-Milgram properties with respect to 'a', it follows by Proposition 2 that V is reflexive. Hence, β is given by an element u of V. Thus, $\exists$ an element $u\neq 0$ such that $(A+B)\,z(u)=0$ $\forall z\in V$, i.e. $a(u, z)+a(z, u)=0$ $\forall z\in V$. But this implies that $u=0$ since 'b' is non-degenerate. This is a contradiction. Hence, $(A+B)(V)$ is dense in V'. Q.E.D.

Reference

[1] S. Ramaswamy: The Lax-Milgram theorem for Banach spaces. I. Proc. Japan Acad., **56**A, 462–464 (1980).