65. A Construction of Lie-Graded Algebras by Graded Generalized Jordan Triples of Second Order

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Introduction. During the last few years the theory of graded algebras and graded triples were developed both in mathematics and physics. In our previous paper [5], from a two dimensional associative triple system W and any generalized Jordan triple system $\mathfrak{J}$ of second order we made a generalized Jordan triple system $W\otimes\mathfrak{J}$ of second order which induced the Lie triple system, and we had a Lie algebra as a standard embedding of the Lie triple system. In this paper we generalize the construction of Lie algebras in [5] to Z- or Z_2-graded case. That is, from the same associative triple system W as in [5] and any graded generalized Jordan triple $\mathfrak{J}$ of second order, we make a graded generalized Jordan triple $W\otimes\mathfrak{J}$ of second order which induces the Lie-graded triple, and we have a Lie-graded algebra as a standard embedding of the induced Lie-graded triple (Theorem 1).

1. Let Δ be Z or Z_2 and let $\mathfrak{B}=\oplus_{i\in\Delta}\mathfrak{B}_i$ be a Δ-graded vector space. Throughout the paper we assume that each vector subspace $\mathfrak{B}_i$ of degree i is finite dimensional and x_i is an element in $\mathfrak{B}_i$. And we also assume that the characteristic of the base field Φ is different from 2 or 3. An endomorphism E_i of $\mathfrak{B}$ is called a graded endomorphism of degree i if $E_i\mathfrak{B}_j\subset\mathfrak{B}_{i+j}$ for all $j\in\Delta$ and the vector space of such endomorphisms is denoted by $\mathrm{End}_i\,\mathfrak{B}$.

Let $\mathfrak{G}=\oplus_{i\in\Delta}\mathfrak{G}_i$ be a Δ-graded vector space with graded bilinear product $[x_i, y_j]_\mp$ satisfying the following conditions:

(1) $$[x_i, y_j]_\mp+(-1)^{ij}[y_j, x_i]_\mp=0,$$

(2) $$(-1)^{ik}[[x_i, y_j]_\mp, z_k]_\mp+(-1)^{ji}[[y_j, z_k]_\mp, x_i]_\mp+(-1)^{kj}[[z_k, x_i]_\mp, y_j]_\mp=0,$$

then $\mathfrak{G}$ is called a Δ-Lie-graded algebra (Δ-LGA) or a Δ-Lie superalgebra (cf. [3], [4], [8]).

2. A Δ-graded vector space $\mathfrak{B}=\oplus_{i\in\Delta}\mathfrak{B}_i$ with a graded trilinear product $\{x_iy_jz_k\}\in\mathfrak{B}_{i+j+k}$ is called a Δ-graded triple (Δ-GT). An endomorphism $D\in\mathrm{End}_i\,\mathfrak{B}$ is called a graded derivation of degree i of $\mathfrak{B}$ if

$$D\{x_jy_kz_l\}=\{Dx_jy_kz_l\}+(-1)^{ij}\{x_jDy_kz_l\}+(-1)^{i(j+k)}\{x_jy_kDz_l\}.$$

Let $\mathrm{Der}_i\,\mathfrak{B}$ be the vector space spanned by these graded derivations of degree i and $\mathrm{Der}\,\mathfrak{B}=\oplus_{i\in\Delta}\mathrm{Der}_i\,\mathfrak{B}$. For any two graded derivations $D_i\in\mathrm{Der}_i\,\mathfrak{B}$, $D_j\in\mathrm{Der}_j\,\mathfrak{B}$ their graded commutator $[D_i, D_j]_\mp=D_iD_j$

$-(-1)^{ij}D_jD_i$ is a graded derivation of degree $i+j$. Hence Der $\mathfrak{V}$ is a Δ-Lie-graded algebra ([10]).

Let $\mathfrak{T}=\oplus_{i\in\Delta}\mathfrak{T}_i$ be a Δ-GT with a product $[x_iy_jz_k]=D(x_i, y_j)z_k$ satisfying the conditions:

(3) $$[x_iy_jz_k]+(-1)^{ij}[y_jx_iz_k]=0,$$

(4) $$(-1)^{ik}[x_iy_jz_k]+(-1)^{ji}[y_jz_kx_i]+(-1)^{kj}[z_kx_iy_j]=0,$$

(5) $$[D(x_i, y_j), D(u_k, v_l)]_{\mp}=D([x_iy_ju_k], v_l)+(-1)^{(i+j)k}D(u_k, [x_iy_jv_l]).$$

Then $\mathfrak{T}$ is called a Δ-Lie-graded triple (Δ-LGT) which is a graded generalization of Lie triple system ([10]). Any Δ-LGA becomes a Δ-LGT with respect to a triple product $[x_iy_jz_k]=[[x_i, y_j]_{\mp}, z_k]_{\mp}$. For a Δ-LGT $\mathfrak{T}=\oplus_{i\in\Delta}\mathfrak{T}_i$ the condition (5) shows that an endomorphism $D(x_i, y_j)$ is a graded derivation of degree $i+j$ of $\mathfrak{T}$ which is called an inner derivation. Let $\text{Inder}_i\,\mathfrak{T}$ be a vector space spanned by inner derivations of degree i in Δ-LGT $\mathfrak{T}$, then $D(\mathfrak{T}, \mathfrak{T})=\oplus_{i\in\Delta}\text{Inder}_i\,\mathfrak{T}$ becomes a Δ-Lie-graded subalgebra of Der $\mathfrak{T}$. This $D(\mathfrak{T}, \mathfrak{T})$ is called a Δ-LGA of graded inner derivations in $\mathfrak{T}$. And the vector space direct sum $D(\mathfrak{T}, \mathfrak{T})\oplus\mathfrak{T}$ becomes a Δ-LGA relative to the following graded product:

$$[D_i+x_i, D_j+y_j]_{\mp}:=[D_i, D_j]_{\mp}+D(x_i, y_j)+D_iy_j-(-1)^{ij}D_jx_i$$

for $D_i\in\text{Inder}_i\,\mathfrak{T}$, $D_j\in\text{Inder}_j\,\mathfrak{T}$, $x_i\in\mathfrak{T}_i$, $y_j\in\mathfrak{T}_j$. This Δ-LGA $D(\mathfrak{T}, \mathfrak{T})\oplus\mathfrak{T}$ is called the standard embedding Δ-LGA of Δ-LGT $\mathfrak{T}$ ([10]).

3. Let W be a two dimensional triple system with product $\{abc\}=l(a, b)c$ which has a basis $\{e_1, e_2\}$ such that $\{e_1e_1e_1\}=\alpha e_1$, $\{e_1e_1e_2\}=\{e_1e_2e_1\}=\{e_2e_1e_1\}=\alpha e_2$, $\{e_1e_2e_2\}=\{e_2e_1e_2\}=\{e_2e_2e_1\}=\beta e_1$, $\{e_2e_2e_2\}=\beta e_2$, where $\alpha, \beta\in\Phi$. Then W is a commutative associative triple system (ATS) (cf. [7]) and is also a Jordan triple system. In the ATS W, we have

(6) $$l(a, b)l(c, d)=l(c, d)l(a, b),$$

(7) $$l(a, b)l(c, d)=l(l(a, b)c, d)=l(c, l(b, a)d).$$

Let $\mathfrak{J}=\oplus_{i\in\Delta}\mathfrak{J}_i$ be a Δ-GT with a product $\{x_iy_jz_k\}$. But $\{x_iy_jz_k\}=L(x_i, y_j)z_k$, and

$$K(x_i, y_j)z_k=(-1)^{jk}\{x_iz_ky_j\}-(-1)^{i(j+k)}\{y_jz_kx_i\}.$$

Then we have

(8) $$[L(x_i, y_j), L(u_k, v_l)]_{\mp}=L(\{x_iy_ju_k\}, v_l)-(-1)^{(i+j)k+ij}L(u_k, \{y_jx_iv_l\}),$$

(9) $$K(K(x_i, y_j)u_k, v_l)=K(x_i, y_j)L(u_k, v_l)+(-1)^{(i+j)(k+l)+kl}L(v_l, u_k)K(x_i, y_j).$$

Then, $\mathfrak{J}$ is called a Δ-graded generalized Jordan triple of second order (Δ-GGJT of 2nd order) which is a graded generalization of a generalized Jordan triple system of 2nd order due to I. L. Kantor ([2], [6], [11]).

Using the identities (6) and (7), we have

Lemma 1. *For the* ATS W *and any* Δ-GGJT $\mathfrak{J}=\oplus_{i\in\Delta}\mathfrak{J}_i$ *of* 2nd *order, define a graded trilinear product in* $W\otimes\mathfrak{J}=\oplus_{i\in\Delta}(W\otimes\mathfrak{J}_i)$ *by*

$$\{a\otimes x_i b\otimes y_j c\otimes z_k\} := \{abc\}\otimes\{x_i y_j z_k\}$$

for $a, b, c\in W$ *and* $x_i\in\mathfrak{J}_i$, $y_j\in\mathfrak{J}_j$, $z_k\in\mathfrak{J}_k$. *Then* $W\otimes\mathfrak{J}$ *becomes a* $\varDelta$-GGJT *of* 2^{nd} *order.*

It is known that a $\varDelta$-GGJT $\mathfrak{J}=\oplus_{i\in\varDelta}\mathfrak{J}_i$ of 2^{nd} order with a product $\{x_i y_j z_k\}$ becomes a $\varDelta$-LGT relative to a new product ([1]):

$$[x_i y_j z_k] := \{x_i y_j z_k\} - (-1)^{ij}\{y_j x_i z_k\} + (-1)^{jk}\{x_i z_k y_j\} - (-1)^{i(j+k)}\{y_j z_k x_i\}.$$

We denote this $\varDelta$-LGT by $\mathfrak{J}^*$ and call an induced $\varDelta$-LGT (from $\mathfrak{J}$). For the $\varDelta$-GGJT $W\otimes\mathfrak{J}$ of 2^{nd} order in Lemma 1, the $\varDelta$-LGT product in $(W\otimes\mathfrak{J})^*$ is as follows: $[a\otimes x_i b\otimes y_j c\otimes z_k] = \{abc\}\otimes[x_i y_j z_k]$ or $D(a\oplus x_i, b\otimes y_j)(c\otimes z_k) = l(a,b)c\otimes D(x_i, y_j)z_k$, where $a, b, c\in W$ and $x_i\in\mathfrak{J}_i$, $y_j\in\mathfrak{J}_j$, $z_k\in\mathfrak{J}_k$. Let $\mathfrak{D}$ be the $\varDelta$-LGA of graded inner derivations $D(a\otimes x_i, b\otimes y_j)$ in the $\varDelta$-LGT $(W\otimes\mathfrak{J})^*$. Then $\mathfrak{S}(W,\mathfrak{J}) = \mathfrak{D}\oplus(W\otimes\mathfrak{J})^*$ is the standard embedding $\varDelta$-LGA of the $\varDelta$-LGT $(W\otimes\mathfrak{J})^*$. By the property of the product in $(W\otimes\mathfrak{J})^*$ we have

$$\mathrm{Inder}_i\,(W\otimes\mathfrak{J})^* = l(W,W)\otimes\mathrm{Inder}_i\,\mathfrak{J}^*,$$

where $l(W,W)$ is the vector space spanned by $\{l(a,b): a, b\in W\}$. If $\alpha\neq 0$ or $\beta\neq 0$ in W, then $\{id_W, l(e_1,e_2)\}$ is a basis of $l(W,W)$, where id_W is the identity endomorphism in W. Hence, we have

$$\mathfrak{D} = id_W\otimes D(\mathfrak{J},\mathfrak{J}) \oplus l(e_1,e_2)\otimes D(\mathfrak{J},\mathfrak{J}),$$

where $D(\mathfrak{J},\mathfrak{J})$ is a $\varDelta$-LGA of graded inner derivations in $\mathfrak{J}^*$.

Then we obtain

Theorem 1. *If* $\alpha\neq 0$ *or* $\beta\neq 0$ *in the* ATS W, *then*

$$\mathfrak{S}(W,\mathfrak{J}) = id_W\otimes D(\mathfrak{J},\mathfrak{J}) \oplus l(e_1,e_2)\otimes D(\mathfrak{J},\mathfrak{J}) \oplus (W\otimes\mathfrak{J})^*$$

is the standard embedding $\varDelta$-LGA *of the* $\varDelta$-LGT $(W\otimes\mathfrak{J})^*$, *and*

$$id_W\otimes D(\mathfrak{J},\mathfrak{J}) \oplus l(e_1,e_2)\otimes D(\mathfrak{J},\mathfrak{J})$$

is a $\varDelta$*-Lie-graded subalgebra of* $\mathfrak{S}(W,\mathfrak{J})$ *satisfying the following graded commutator relations:*

$$[\mathfrak{L},\mathfrak{L}]_{\mp}\subset\mathfrak{L},\quad [\mathfrak{M},\mathfrak{M}]_{\mp}\subset\mathfrak{L},\quad [\mathfrak{L},\mathfrak{M}]_{\mp}\subset\mathfrak{M},$$

where $\mathfrak{L} = id_W\otimes D(\mathfrak{J},\mathfrak{J})$, $\mathfrak{M} = l(e_1,e_2)\otimes D(\mathfrak{J},\mathfrak{J})$.

4. Let $\mathfrak{J}=\oplus_{i\in\varDelta}\mathfrak{J}_i$ be a $\varDelta$-GGJT of 2^{nd} order. Now we consider the vector space direct sum $\mathfrak{J}\oplus\mathfrak{J} = \oplus_{i\in\varDelta}(\mathfrak{J}_i\oplus\mathfrak{J}_i)$, which is spanned by $\{x_i\oplus\bar{x}_i : x_i, \bar{x}_i\in\mathfrak{J}_i, i\in\varDelta\}$. Then we denote an element $x_i\oplus\bar{x}_i$ in $\mathfrak{J}\oplus\mathfrak{J}$ by $\begin{pmatrix} x_i \\ \bar{x}_i\end{pmatrix}$ and define a triple product in $\mathfrak{J}\oplus\mathfrak{J}$ by

$$\left\{\begin{pmatrix} x_i \\ \bar{x}_i\end{pmatrix}\begin{pmatrix} y_j \\ \bar{y}_j\end{pmatrix}\begin{pmatrix} z_k \\ \bar{z}_k\end{pmatrix}\right\} := \begin{pmatrix} \alpha\{x_i y_j z_k\} + \beta\{x_i\bar{y}_j\bar{z}_k\} + \varepsilon\beta\{\bar{x}_i y_j \bar{z}_k\} + \beta\{\bar{x}_i\bar{y}_j z_k\} \\ \alpha\{x_i y_j \bar{z}_k\} + \varepsilon\alpha\{x_i\bar{y}_j z_k\} + \alpha\{\bar{x}_i y_j z_k\} + \beta\{\bar{x}_i\bar{y}_j\bar{z}_k\}\end{pmatrix}, \tag{10}$$

where α, β are the elements of the base field Φ and $\varepsilon=\pm 1$. Then the product defined above is a graded triple product in $\mathfrak{J}\oplus\mathfrak{J}$. By straightforward calculations, we have

Theorem 2. *Let* $\mathfrak{J}$ *be a* $\varDelta$-GGJT *of* 2^{nd} *order, then* $\mathfrak{J}\oplus\mathfrak{J}$ *becomes* *a Δ-GGJT of 2^{nd} order with respect to the product defined above.*

The Δ-GGJT of 2^{nd} order obtained in Theorem 2 is denoted by $(\mathfrak{J}\oplus\mathfrak{J})_\varepsilon$. For $\varepsilon=+1$, if we define a linear mapping f of $W\otimes\mathfrak{J}$ into $(\mathfrak{J}\oplus\mathfrak{J})_{+1}$ by $f(e_1\otimes x_i+e_2\otimes \bar{x}_i)=\begin{pmatrix}x_i\\ \bar{x}_i\end{pmatrix}$ for all $i\in\Delta$, we have the following

Theorem 3. *$W\otimes\mathfrak{J}$ is isomorphic to $(\mathfrak{J}\oplus\mathfrak{J})_{+1}$ as Δ-GGJT of 2^{nd} order.*

By direct calculations, we see that the product in the induced Δ-LGT $(\mathfrak{J}\oplus\mathfrak{J})^*_\varepsilon$ is given as follows

$$(11)\qquad \left[\begin{pmatrix}x_i\\ \bar{x}_i\end{pmatrix}\begin{pmatrix}y_j\\ \bar{y}_j\end{pmatrix}\begin{pmatrix}z_k\\ \bar{z}_k\end{pmatrix}\right]=\begin{pmatrix}\alpha[x_iy_jz_k]+\beta[x_i\bar{y}_j\bar{z}_k]+\varepsilon\beta[\bar{x}_iy_j\bar{z}_k]+\beta[\bar{x}_i\bar{y}_jz_k]\\ \alpha[x_iy_j\bar{z}_k]+\varepsilon\alpha[x_i\bar{y}_jz_k]+\alpha[\bar{x}_iy_jz_k]+\beta[\bar{x}_i\bar{y}_j\bar{z}_k]\end{pmatrix},$$

where $[x_iy_jz_k]$ is the product in $\mathfrak{J}^*$.

Remark 1. If we put $\varepsilon=-1$ in (10), $(\mathfrak{J}\oplus\mathfrak{J})_{-1}$ is isomorphic to $J(\alpha,\beta,0)$ in [1]. Hence Δ-LGA can be constructed by $(\mathfrak{J}\oplus\mathfrak{J})_{-1}$ as in [1].

For an induced Δ-LGT $\mathfrak{J}^*$, we consider the vector space direct sum $\mathfrak{J}^*\oplus\mathfrak{J}^*$, which is spanned by $\{x_i\oplus\bar{x}_i : x_i, \bar{x}_i\in\mathfrak{J}^*_i, i\in\Delta\}$. Then, we denote an element $x_i\oplus\bar{x}_i$ in $\mathfrak{J}^*\oplus\mathfrak{J}^*$ by $\begin{pmatrix}x_i\\ \bar{x}_i\end{pmatrix}$ and define a triple product $\mathfrak{J}^*\oplus\mathfrak{J}^*$ by

$$(12)\qquad \left[\begin{pmatrix}x_i\\ \bar{x}_i\end{pmatrix}\begin{pmatrix}y_j\\ \bar{y}_j\end{pmatrix}\begin{pmatrix}z_k\\ \bar{z}_k\end{pmatrix}\right]=\begin{pmatrix}\alpha[x_iy_jz_k]+\beta[x_i\bar{y}_j\bar{z}_k]+\beta[\bar{x}_iy_j\bar{z}_k]+\beta[\bar{x}_i\bar{y}_jz_k]\\ \alpha[x_iy_j\bar{z}_k]+\alpha[x_i\bar{y}_jz_k]+\alpha[\bar{x}_iy_jz_k]+\beta[\bar{x}_i\bar{y}_j\bar{z}_k]\end{pmatrix}.$$

Then, using the expression (11) we have

Theorem 4. *$\mathfrak{J}^*\oplus\mathfrak{J}^*$ becomes a Δ-LGT and is isomorphic to $(\mathfrak{J}\oplus\mathfrak{J})^*_{+1}$ as Δ-LGT.*

Remark 2. If we put $\alpha=1$ and $\beta=0, \pm 1$ in the graded triple product (12), we get a graded generalization of the Lie triple product defined by Y. Taniguchi (cf. [9]).

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