62. On Ranked Linear Spaces. I

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§ 1. **Introduction.** Since the notion of ranked spaces was introduced by Kunugi [1], many authors have developed the theory of these spaces. In particular Okano [8] and Nakanishi [7] discussed the completion of these spaces and Washihara [10] gave a definition of ranked linear spaces. On the other hand, Okano [9] and Nagakura [5] utilized completion of some concretely given ranked linear spaces.

It is to be noticed, however, that the method of completion given in [7], [8] can not be applied to the ranked linear spaces as defined in [10].

In the present note, we give another definition of ranked linear spaces which can be, as we shall show in a forthcoming note, completed by a method suggested in [1] and which will cover the method in [5], [9]. (We see easily that our definition of ranked linear spaces is narrower than that given in [10].)

We shall give furthermore a formulation of Closed graph theorem and Banach-Steinhaus theorem in our spaces, which seems natural to us.

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§ 2. **Definition of ranked linear spaces.** In the following we limit our considerations to ranked spaces of indicator ω_0 (in the terminology of [2]) over a linear space E over real or complex field.

For each non-negative integer n and for each point p of E we have a family $\mathfrak{V}_n(p)$ of subsets of E containing p, called *pre-neighborhoods* of p of rank n, which will be generally denoted by $V(p,n)$. It is required that for every $V(p,n)$ and $n'\geq n$ there is a $V(p,n')\subset V(p,n)$.

For $\mathfrak{V}_n(0)$ (0 being the origin of E), the following is postulated:

(1) $V(0,n)$ is symmetrical and E itself is a pre-neighborhood of 0 of rank 0.

(2) If $n<m$ and $V\subset U$ where $U\in\mathfrak{V}_n(0)$ and $V\in\mathfrak{V}_m(0)$, then $V+V\subset U$ and $\alpha V\subset U$ for any scalar α such that $|\alpha|\leq 1$.

(3) Let n_i, m_i $(i=0,1,\cdots)$ be increasing sequences of non-negative integers such that $n_i<n_{i+1}$, $m_i<m_{i+1}$ and U_i, V_i be sequences of pre-neighborhoods of 0 of rank n_i, m_i respectively such that $U_i\supset U_{i+1}$, $V_i\supset V_{i+1}$. For a pre-neighborhood W of 0 of rank l such that $W\supset U_0+V_0$

there exist $l'>l$, $W'\in\mathfrak{V}_{l'}(0)$ and some integer n satisfying $W\supset W'\supset U_n+V_n$.

(4) If $U\in\mathfrak{V}_n(0)$, $V\in\mathfrak{V}_m(0)$, $V\subset U$ and $n<k<m$, then there is a $W\in\mathfrak{V}_k(0)$ such that $U\supset W\supset V$.

Next, suppose that the family $\mathfrak{V}(p)$ of pre-neighborhoods of each point p of E is given as follows:

$$\mathfrak{V}(p)=\bigcup_{n=0}^{\infty}\mathfrak{V}_n(p),\qquad \mathfrak{V}_n(p)=\{p+V(0,n):\ V(0,n)\in\mathfrak{V}_n^{(p)}(0)\},$$

where $\mathfrak{V}_n^{(p)}(0)$ is a non-empty subfamily of $\mathfrak{V}_n(0)$. And also suppose that the family $\{\mathfrak{V}^{(p)}(0)=\bigcup_{n=0}^{\infty}\mathfrak{V}_n^{(p)}(0): p\in E\}$ has the following properties (I)–(III):

(I)(i) If $U\in\mathfrak{V}(0)$ and $p\in U$, then $U\in\mathfrak{V}^{(p)}(0)$.

(ii) If $U, V\in\mathfrak{V}(0)$, $V\subset U$ and $V\in\mathfrak{V}^{(p)}(0)$, then $U\in\mathfrak{V}^{(p)}(0)$.

(iii) For any $U\in\mathfrak{V}_n^{(p)}(0)$, there is a $V\in\mathfrak{V}_{n+1}^{(p)}(0)$ such that $V\subset U$.

(II)(i) If $W\in\mathfrak{V}_l^{(p)}(0)\cap\mathfrak{V}_l^{(q)}(0)$, $W'\in\mathfrak{V}_{l'}(0)$, $W\subset W'$ and $l'<l$ then $W'\in\mathfrak{V}_{l'}^{(p+q)}(0)$.

(ii) If $W\in\mathfrak{V}_n^{(p)}(0)$, then $W\in\mathfrak{V}_n^{(\alpha p)}(0)$ for every scalar $\alpha\neq 0$.

(III) If $U\in\mathfrak{V}_n^{(p)}(0)$, $V\in\mathfrak{V}_m^{(p)}(0)$, $W\in\mathfrak{V}_l^{(r)}(0)$, $p+U\supset p+V\supset r+W$ and $n<m<l$ then $V+W\subset U$.

Remark. The first half of (2), (3) and (III) can be replaced with the following stronger condition:

If $U\in\mathfrak{V}_n(0)$, $V_1\in\mathfrak{V}_{m_1}(0)$, $V_2\in\mathfrak{V}_{m_2}(0)$, $U\supset V_1\cup V_2$ and $n<\min(m_1, m_2)$, then $V_1+V_2\subset U$ and there is a $U'\in\mathfrak{V}_{\min(m_1,m_2)-1}(0)$ such that $U\supset U'\supset V_1\cup V_2$.

Now we give the following

Definition. We call $(E, \mathfrak{V}_n, \mathfrak{V}_n^{(p)})$ a *ranked linear space* when these axioms (1)–(4), (I)–(III) are satisfied.

We notice that the ranked linear spaces thus defined differ from topological vector spaces or ranked linear spaces in [5], [9] in the following point; the family of pre-neighborhoods of each point p is obtained by translating a subfamily of the family of pre-neighborhoods of 0 which depends on p, and the family of pre-neighborhoods of 0 is not necessarily filtrant.

Example. Give a sequence of normed spaces $\{(E_k, \|\cdot\|_k)\}_{k=1,2,\dots}$ with $E_1\subsetneqq E_2\subsetneqq\cdots$ and $\|\cdot\|_k\geq\|\cdot\|_{k+1}$ on E_k. Let $E=\bigcup_k E_k$ be its *ranked union space* (cf. [6]) where $\mathfrak{V}_n^{(p)}(0)=\{V(0,k,n)=\{q\in E_k: \|q\|_k<1/2^n\}: p\in E_k\}$. Then E becomes a ranked linear space defined in this section.

Remark. We may consider the structure of ranked linear space on E and its completion $\hat{E}$ as in [5], [9], but in that case we do not have $\hat{E}=\bigcup_k\hat{E}_k$. We shall show in a later note another natural way to complete these spaces so that this takes place.

§3. Basic notions. We remind some notions from [2] adapted to our case.

A sequence of pre-neighborhoods $u=\{p_i+U_i\}_{i=0,1,\dots}$ such that $p_i+U_i\supset p_{i+1}+U_{i+1}$ is said to be *fundamental* if there exists a sequence $\{n(i)\}$ of integers such that $n(i)\le n(i+1)$, $U_i\in\mathfrak{B}_{n(i)}^{(p_i)}(0)$ satisfying: For every i there is a j such that $i\le j$, $p_j=p_{j+1}$ and $n(j)<n(j+1)$.

Such u is said to be a *p-fundamental sequence*, if $p_i=p$ for all i.

A fundamental sequence is called *canonical*, if $p_{2i}=p_{2i+1}$ and $n(2i)<n(2i+1)$ for all i.

A ranked linear space E is said to be *separated* when for any 0-fundamental sequence $u=\{U_i\}$, we have $\theta(u)=\bigcap_{i=0}^{\infty}U_i=\{0\}$.

We say that E is *complete*, if every fundamental sequence has non-empty intersection.

For a sequence $\{p_i\}$ of points of E, it is said to *converge* to $p\in E$ if there exists a p-fundamental sequence $\{p+U_i\}$ such that for each i we have a $J(i)$ satisfying $p_j\in p+U_i$ for all $j\ge J$.

Let $(E,\mathfrak{B}_{E,n},\mathfrak{B}_{E,n}^{(p)})$ and $(F,\mathfrak{B}_{F,n},\mathfrak{B}_{F,n}^{(q)})$ be ranked linear spaces and let f be a mapping from E into F. Then f is said to be *continuous* at $p\in E$ if for any 0-fundamental sequence $\{U_i\}$ in E such that $U_i\in\mathfrak{B}_E^{(p)}(0)$ there exists a 0-fundamental sequence $\{V_i\}$ in F such that $V_i\in\mathfrak{B}_F^{(f(p))}(0)$ and $\{f(p+U_i)\}\prec\{f(p)+V_i\}$.

If f is linear and continuous at 0, then f is continuous at every point of E. (This is easy to see from the property (3) of $\mathfrak{B}(0)$ and the properties (I)(ii), (iii) of $\mathfrak{B}^{(p)}(0)$.)

(Here, for two decreasing sequences of subsets $\{A_i\}$ and $\{B_i\}$, we denote $\{A_i\}\prec\{B_i\}$ when for every B_i there is an A_j such that $A_j\subset B_i$ and denote $\{A_i\}\sim\{B_i\}$ when $\{A_i\}\prec\{B_i\}$ and $\{B_i\}\prec\{A_i\}$.)

We consider the product linear space $G=E\times F$ and define the system of pre-neighborhoods of $r=(p,q)\in G$ as follows:

$$\mathfrak{B}_G^{(r)}(0)=\bigcup_{n=0}^{\infty}\mathfrak{B}_{G,n}^{(r)}(0),\qquad \mathfrak{B}_{G,n}^{(r)}(0)=\{V_1\times V_2: V_1\in\mathfrak{B}_{E,n}^{(p)}(0),\ V_2\in\mathfrak{B}_{F,n}^{(q)}(0)\}.$$

This $(G,\mathfrak{B}_{G,n},\mathfrak{B}_{G,n}^{(r)})$ is called the *product ranked linear space* of

$$(E,\mathfrak{B}_{E,n},\mathfrak{B}_{E,n}^{(p)})\quad\text{and}\quad(F,\mathfrak{B}_{F,n},\mathfrak{B}_{F,n}^{(q)}).$$

For two ranked linear spaces E and F, these are said to be *isomorphic* or *equivalent* if there is a one-to-one, linear mapping τ from E onto F such that for any fundamental sequence $\{p_i+U_i\}$ in E there exists a fundamental sequence $\{q_i+V_i\}$ in F satisfying $\{\tau(p_i+U_i)\}\sim\{q_i+V_i\}$, and conversely for any fundamental sequence $\{q_i+V_i\}$ in F there exists a fundamental sequence $\{p_i+U_i\}$ in E satisfying

$$\{\tau^{-1}(q_i+V_i)\}\sim\{p_i+U_i\}.$$

Let $(E,\mathfrak{B}_{E,n},\mathfrak{B}_{E,n}^{(p)})$ and $(F,\mathfrak{B}_{F,n},\mathfrak{B}_{F,n}^{(q)})$ be ranked linear spaces and suppose that E is a linear subspace of F. E is said to be a *ranked linear subspace* of F, if the following holds:

For each $p\in E$, $\mathfrak{B}_{E,n}^{(p)}(0)=\{V\cap E: V\in\mathfrak{B}_{F,n}^{(p)}(0)\}$ and

$$\mathfrak{B}_{F,n}^{(p)}(0)=\{V\in\mathfrak{B}_F(0): V\cap E\in\mathfrak{B}_{E,n}^{(p)}(0)\},$$

furthermore, if $U_1(0, n_1), U_2(0, n_2) \in \mathfrak{B}_E(0)$, $V_1 \in \mathfrak{B}_F(0)$, $U_1 = V_1 \cap E$, $U_1 \supset U_2$ and $n_1 < n_2$, then there exists a $V_2 \in \mathfrak{B}_F(0)$ such that $V_1 \supset V_2$ and $U_2 = V_2 \cap E$.

It is easily seen that if a space E is a linear subspace of a ranked linear space $(F, \mathfrak{B}_{F,n}, \mathfrak{B}_{F,n}^{(q)})$ and there is a system of families of subsets of E, $(\mathfrak{B}_{E,n}, \mathfrak{B}_{E,n}^{(p)})$, which satisfies the above conditions, then $(E, \mathfrak{B}_{E,n}, \mathfrak{B}_{E,n}^{(p)})$ becomes a ranked linear space.

Let A be a subset of E. A is said to be *dense* in E if for every $p \in E$ and every pre-neighborhood $p+U$ in E, we have $(p+U) \cap A \neq \phi$.

§4. **Closed graph theorem and Banach-Steinhause theorem.** First we introduce the following definitions.

Definition 1. Let $\mathfrak{F}_0$ be a family of 0-canonical fundamental sequences. When for any 0-fundamental sequence u_0, there is a $v_0 \in \mathfrak{F}_0$ such that $u_0 \prec v_0$, we say that the family $\mathfrak{F}_0$ is *basic*.

In this section we shall consider only the case that E has a basic family, denoted by $\mathfrak{F}_0^E$, such that for every $\{U_i\} \in \mathfrak{F}_0^E$ each point of U_0 is absorbed by all U_i.

Definition 2. We say that E is *absorbent* if $E \subset \cup \{U_0 : \{U_i\} \in \mathfrak{F}_0^E\}$.

Definition 3. Let A be a subset of E. We define as the *closure* of A in E, the set such that $\{p \in E$: there is a $\{U_i\} \in \mathfrak{F}_0^E$ such that $U_i \in \mathfrak{B}^{(p)}(0)$ for all i and $(p+U_i) \cap A \neq \phi$ for all $i\}$. We denote this closure of A by $\bar{A}$. A is said to be *closed* if $\bar{A} = A$.

Definition 4. Let A be a subset of E. A is said to be *bounded* in E if there exist $\{U_i\} \in \mathfrak{F}_0^E$ and $\{\alpha_i > 0\}$ such that $A \subset \alpha_i U_i$ for all i.

Definition 5. A ranked linear space E is said to be *s-space* when we can take a basic family $\mathfrak{F}_0^E$ such that for every $\{V_i\} \in \mathfrak{F}_0^E$ there exists a $\{U_i\} \in \mathfrak{F}_0^E$ such that $\{\bar{V}_i\} \prec \{U_i\}$.

Now, Closed graph theorem and Banach-Steinhaus theorem are formulated as follows. Their proof runs just as in [3], [4].

Theorem 1 (Closed graph theorem). *Let $(E, \mathfrak{B}_{E,n}, \mathfrak{B}_{E,n}^{(p)})$ and $(F, \mathfrak{B}_{F,n}, \mathfrak{B}_{F,n}^{(q)})$ be complete ranked linear spaces and suppose that $(F, \mathfrak{B}_{F,n}, \mathfrak{B}_{F,n}^{(q)})$ satisfies the following condition: For any $V \in \mathfrak{B}_{F,n}(0)$ there exists a family of countable pre-neighborhoods of 0 in F with rank $n+1$ such that each of these pre-neighborhoods is contained in V and its union absorbs each point of V. If f is a closed linear mapping from E into F, then f is continuous at every point of E.* (Here f is said to be closed if the set $\{(p, f(p)) : p \in E\}$ is closed in $E \times F$.)

Theorem 2 (Banach-Steinhaus theorem). *Let E be complete and F be an s-space such that the set of terms of fundamental sequences in $\mathfrak{F}_0^F$ is countable. Let A be a family of continuous linear mappings from E into F. If A is pointwise bounded, then A is equi-continuous at every point of E.* (Here A is said to be pointwise bounded if for each point $p \in E$ the set $\{f(p) : f \in \mathsf{A}\}$ is bounded in F. A is said to be equicontinuous at $p \in E$ if for every $\{U_i\} \in \mathfrak{F}_0^E$ such that $U_i \in \mathfrak{V}_E^{(p)}(0)$, there exists a $\{V_i\} \in \mathfrak{F}_0^F$ such that $V_i \in \mathfrak{V}_F^{(f(p))}(0)$ and $\{f(p+U_i)\} \prec \{f(p)+V_i\}$ for all $f \in \mathsf{A}$.)

References

[1] K. Kunugi: Sur les espaces complets et régulièrement complets. I, II, III. Proc. Japan Acad., **30**, 553–556 (1954); **30**, 912–916 (1954); **31**, 49–53 (1955).

[2] ——: Sur la méthode des espaces rangés. I, II. ibid., **42**, 318–322 (1966); **42**, 549–554 (1966).

[3] Y. Nagakura: On closed graph theorem. ibid., **48**, 665–668 (1972).

[4] ——: On Banach-Steinhaus theorem. ibid., **49**, 333–336 (1973).

[5] ——: The nuclear spaces introduced by some family of operators in Hilbert space. I, II. TRU Math., Sci. Univ. of Tokyo, **13**(1), 9–24 (1977); **13**(2), 13–26 (1977).

[6] S. Nakanishi: On the strict union of ranked metric spaces. Proc. Japan Acad., **50**, 603–607 (1974).

[7] ——: Completions of ranked spaces. I, II. Math. Japonica, **23**, 409–422 (1978); **23**, 629–646 (1978).

[8] H. Okano: On the completion of ranked spaces. Proc. Japan Acad., **33**, 338–340 (1957).

[9] ——: Sur les intégrals (E.R.) et ses applications. Osaka Math. J., **11**, 187–212 (1959).

[10] M. Washihara: On ranked spaces and linearity. I, II. Proc. Japan Acad., **43**, 584–589 (1967); **45**, 238–242 (1969).