5. *Bochner Problem on a Topological Vector Space with a Quasi-Invariant Measure*

By Yoshiaki OKAZAKI
Department of Mathematics, Kyushu University

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1. Let F be a locally convex space and ρ be another vector topology on F. Then ρ is called *admissible* if for every positive definite function ϕ, the ρ-continuity of ϕ is equivalent to the existence of a $\sigma(F', F)$-Radon measure on F' (the topological dual) with the Fourier transform ϕ. The admissible topologies are not unique even on the Hilbert space (cf., Sazonov [9] and Gross [1]). We say the weakest of all admissible topologies (if exists) *the S-topology*. In case of a Hilbert space and a nuclear Fréchet space, the admissible topologies were given by Sazonov [9] and Minlos [6], respectively. Mouchtari [7] proved that if F is a Banach space with the metric approximation property and F' is L^0-embeddable, then the S-topology exists, and gave explicitly the S-topology on L^p $(2\leqq p<\infty)$. The first aim of this paper is to prove that if F is a Banach space of dual $\mathcal{S}_p$ $(1<p\leqq 2)$, then the S-topology exists and is given by a family of absolutely summing operators (Theorem 1).

Let ϕ be a continuous positive definite function on a locally convex space F. Then, in general, a σ-additive measure with the Fourier transform ϕ does not exist on F'. Sazonov [9] proved that, in the case where F is a Hilbert space, a σ-additive measure with the Fourier transform ϕ exists on a Hilbert-Schmidt extension of F'. The second aim of this paper is to find a suitable extension G of F' such that for every ϕ, there exists a σ-additive measure on G with the Fourier transform ϕ.

Let E be a locally convex space of second category (or barrelled), F be a complete locally convex space with the metric approximation property and $\iota : E\to F$ be a continuous injection. We shall show that if there exists an $\iota(E)$-quasi-invariant Radon measure on F, then for every continuous positive definite function ϕ on F, a $\sigma(E', E)$-Radon measure with the Fourier transform $\phi\circ\iota$ exists on E' (Theorem 3). The fundamental tool to prove Theorem 3 is a generalization of Xia's inequality (Xia [11], also Koshi and Takahashi [1]).

2. Let $FD(F)$ be the set of all finite-dimensional subspaces of F and $W^0=\{x'\in F'; \langle x, x'\rangle=0$ for every $x\in W\}$ for $W\in FD(F)$. The finite-dimensional space F'/W^0 has the natural Borel field $\mathcal{B}(W)$. We

denote by $Z(F', F)=\bigcup\{\Pi_W^{-1}\mathcal{B}(W);\ W\in FD(F)\}$, where $\Pi_W: F'\to F'/W^0$ is the projection. *A cylinder set measure* ν on F' is a non-negative finitely additive set function on $Z(F', F)$ with $\nu(F')=1$ which is σ-additive on $\Pi_W^{-1}(W)$ for each W in $FD(F)$. *The Fourier transform* of ν is defined by $\nu^{\wedge}(x)=\int_{F'}\exp(i\langle x, x'\rangle)d\nu(x')$. For a continuous positive definite function ϕ on F, there is a cylinder set measure ν on $Z(F', F)$ with $\nu^{\wedge}(x)=\phi(x)$ as is well known.

We say a Banach space E is *of dual* $\mathcal{S}_p$ $(1<p\leqq 2)$ if the dual E' is isomorphic to a subspace of $L^p(\mu)$, where μ is a positive measure. A cylinder set measure ν on the dual Banach space E' is called *of type* p $(0<p<\infty)$ provided there is $C>0$ such that

$$\left(\int_{E'}|\langle x, x'\rangle|^p d\nu(x')\right)^{1/p}\leqq C\|x\| \qquad \text{for every } x\in E.$$

Let E, F be Banach spaces and $u: E\to F$ be a continuous linear operator. Then u is called *absolutely p-summing* $(0<p<\infty)$ if there is $C>0$ such that $\sum_{i=1}^n\|u(x_i)\|^p\leqq C\sup\{\sum_{i=1}^n|\langle x_i, x'\rangle|^p;\ \|x'\|\leqq 1\}$, for every finite $x_1,\cdots,x_n\in E$. In case $p=1$, we say *u an absolutely summing operator*. The next lemma is a particular case of Maurey [4, Théorème 2].

Lemma 1. *Let E be a Banach space of dual $\mathcal{S}_p$ $(1<p\leqq 2)$ and q be $1<q\leqq 2$. Then for a continuous linear mapping $R: E\to L^q([0,1], dx)$, dx is the Lebesgue measure, the following conditions are equivalent.*

(1) *R is absolutely summing,*

(2) *for every cylinder set measure λ of type 1 on $(L^q[0,1])'$, the image $R'(\lambda)$ is $\sigma(E', E)$-Radon with $\int\|x'\|dR'(\lambda)(x')<\infty$, and*

(3) *$R'(\lambda_q)$ is $\sigma(E', E)$-Radon with $\int\|x'\|dR'(\lambda_q)(x')<\infty$, where $\lambda_q^{\wedge}(f)=\exp(-|f|_{L^q}^q)$ for each $f\in L^q[0,1]$.*

Theorem 1. *Let E be a Banach space of dual $\mathcal{S}_p$ $(1<p\leqq 2)$. Then the family of seminorms* $(*)$ $x\to\|R(x)\|_{L^q[0,1]}$ *determines the S-topology, where q is arbitrary but fixed as $1<q<p$ and R varies over all absolutely summing operators of E into $L^q[0,1]$.*

Proof. Let τ be the vector topology determined by the seminorms $(*)$. Let ϕ be a τ-continuous positive definite function and $T: E\to L^0(E^a, \boldsymbol{P})$ be $T(x)=\langle x,\ \rangle$ where $\boldsymbol{P}$ is a σ-additive measure on E^a with the Fourier transform ϕ. By the τ-continuity, there is $\delta_n>0$ and an absolutely summing operator $R_n: E\to L^q[0,1]$ such that $\|R_n(x)\|<\delta_n$ implies that $\boldsymbol{P}(x'\in E^a;\ |T(x)(x')|>1)<1/n$. Fix s as $1<s<q$. Then by the Nikishin-Maurey's theorem, see Maurey [5, Proposition 4], E^a can be divided into a disjoint union of A_n such that the mapping $T_n(x)=T(x)\chi_{A_n}$ is absolutely summing from E into $L^s(A_n, \boldsymbol{P}|A_n)$. By Lemma

1, the measure ν_n determined by $\nu_n{}^\wedge(x)=\int_{A_n}\exp(i\langle x, x'\rangle)dP(x')$ is $\sigma(E', E)$-Radon since ν_n is the image $T'_n(\lambda)$, where

$$\lambda^\wedge(f)=\int \exp(if(x'))dP(x'),\ f\in L^s(E^a, \boldsymbol{P}).$$

Hence $\nu=\sum_{n=1}^{\infty}\nu_n$ is $\sigma(E', E)$-Radon and $\nu^\wedge(x)=\phi(x)$.

Conversely, if ν is $\sigma(E', E)$-Radon with $\int\|x'\|^q d\nu(x')<\infty$, then the positive definite function $\exp\left(-\int|\langle x, x'\rangle|^q d\nu(x')\right)$ defines a Radon measure λ on E', see Mouchtari [7]. Thus $\nu^\wedge(x)$ is continuous by the seminorm $\|R(x)\|_{L^q(\lambda)}$, where $R: E\to L^q(E', \lambda)$ be $R(x)=\langle x,\ \rangle$. Since $L^q(\lambda)$ is isometric to $L^q[0, 1]$, $\nu^\wedge(x)$ is continuous with respect to τ.

By (3) in Lemma 1, τ is the weakest one of all admissible topologies. This proves the assertion.

3. Let E, F be complete locally convex spaces and $\iota: E\to F$ be a continuous injection. Suppose that there is an $\iota(E)$-quasi-invariant Radon measure μ on F. That is, $\mu_x\sim\mu$ (equivalent) for every $x\in\iota(E)$, where $\mu_x(A)=\mu(A-x)$. Let K be a compact convex subset with $\mu(K)>0$ and put $L=2K$. Let p be $0<p<\infty$.

Lemma 2. *Let x_0 be in $\iota(E)$. Then there exists $C>0$ such that* $|\langle x_0, x'\rangle|\leqq C\left(\int_L|\langle y, x'\rangle|^p d\mu(y)\right)^{1/p}$ *for every $x'\in F'$.*

Proof. Suppose that $\int_L|\langle y, x'_n\rangle|^p d\mu(y)\to 0$ and $\langle x_0, x'_n\rangle=1$ for $x'_n\in F'$. Taking a subsequence we may assume that $\langle y, x'_n\rangle\to 0$ μ-a.e. on L. By the quasi-invariance, it holds that $\iota(E)\subset\bigcup_{n=1}^{\infty}nK$, so $\delta x_0\in K$ for small $\delta>0$. Put $L_0=\{y\in L;\ \langle y, x'_n\rangle\to 0\}$, then $\mu(L\cap L_0^C)=0$. Thus we have $\mu(L_0\cap(L+\delta x_0))\geqq\mu(K)>0$. Consequently, it must be $\mu_{\delta x_0}(L_0\cap(L+\delta x_0))>0$. But it follows that $\mu_{\delta x_0}(L_0\cap(L+\delta x_0))=\mu((L_0-\delta x_0)\cap L)=\mu((L_0-\delta x_0)\cap L_0)=0$, which is a contradiction.

Theorem 2 (Xia [11], Koshi and Takahashi [2]). *Suppose that E is of second category or a barrelled space. Then there exists a neighborhood V of 0 in E such that for all $x'\in F'$*

$$(**)\qquad \sup_{x\in V}|\langle\iota(x), x'\rangle|\leqq\left(\int_L|\langle y, x'\rangle|^p d\mu(x)\right)^{1/p}.$$

Proof. Put $V=\{x\in E;\ (**)\text{ holds}\}$. Then V is closed convex balanced subset. By Lemma 2, we have $E=\bigcup_{n=1}^{\infty}nV$, which implies the assertion.

By this theorem, ι' can be decomposed as $F'_\tau\to S(\infty)\to S(p)\to E'_{V^0}$, where $S(\infty)$ (resp. $S(p)$) is a closed subspace of $L^\infty(L, \mu|L)$ (resp. $L^p(L, \mu|L)$), τ is the Mackey topology and $E'_{V^0}=\bigcup_{n=1}^{\infty}nV^0$ is the Banach space with the unit ball $V^0=\{z'\in E';\ |\langle z, z'\rangle|\leqq 1\text{ for every } z\in V\}$. Here we remark that the mapping $S(\infty)\to E'_{V^0}$ is absolutely p-summing by the Pietsch's theorem, see Pietsch [8].

We say a locally convex space F has *the metric approximation property* if there is a net $\{T_\alpha\}$ of linear operators of E into E with the finite-dimensional ranges such that $\{T_\alpha\}$ is equicontinuous and $T_\alpha x \to x$ in E.

Now applying the general theory of absolutely p-summing operators to the factorization of $\iota': F'_\tau \to S(\infty) \to S(p) \to E'_{V0}$, in particular the case $0<p<1$ (see Maurey [3] and Schwartz [10]), we have the following Bochner's theorem.

Theorem 3. *Let E, F be complete locally convex spaces and $\iota: E \to F$ be a continuous injection. Suppose that there is an $\iota(E)$-quasi-invariant Radon measure on F. Suppose also that E is of second category (or barrelled) and F has the metric approximation property. Then for every continuous positive definite function ϕ on F, there exists a $\sigma(E', E)$-Radon measure with the Fourier transform $\phi \circ \iota$.*

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