56. A Note on the Fundamental Theorem of Calculus

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The entitled theorem is sometimes called as

The criteria for monotonicity. It reads: *Let $y=f(x)$ be a real-valued continuous function defined on a closed interval $[a,b]$. If the derivative $f'(x)$ exists and >0 for all x of the open interval (a,b), then, for all c,d with $a<c<d<b$, we must have $f(c)<f(d)$.*

To this theorem, the present author should like to propose a proof which does not appeal to the *Mean Value Theorem* (cf. L. Bers [1], 223–224 and P. Lax-S. Burstein-A. Lax [2], 103) and which also gives the proof of the *Intermediate Value Theorem.*

Proof. Assume the contrary and let $f(c)\geqq f(d)$. Since $f'(c)>0$, there exists e with $c<e<d$ and $f(e)>f(d)$. Let m be any number satisfying $f(e)>m>f(d)$. Consider the graph $G(f;e,d)$ of f starting from the point $\{e,f(e)\}$ and ranging towards the point $\{d,f(d)\}$. Take *the first encounter point* $\{g,f(g)\}$ of the graph $G(f;e,d)$ with the line $y=m$. The existence of such point $\{g,f(g)\}$ is proved as follows.

Let S be the set of all points $x_1\in[e,d]$ satisfying the condition that $f(x)>m$ for all $x\in[e,x_1]$. Let x_∞ be the least upper bound of the set S. Then $x_\infty \overline{\in} S$. If otherwise, $f(x_\infty)>m$ so that, by the continuity of f, $f(x)>m$ for all x sufficiently close to x_∞. Hence there should exist a point $x_1\in S$ which is to the right of x_∞. This is absurd. Hence $x_\infty \overline{\in} S$ is a limit point of the set S and so $f(x_\infty)=m$. Therefore, $e<x_\infty<d$ and we can take x_∞ for g.

Since $\{g,f(g)\}$ is the first encounter point of the graph $G(f;e,d)$ with the line $y=m$, we must have $f(x)>m$ for all x with $e\leqq x<g$. This implies, by $f(g)=m$, that $f'(g)\leqq 0$, contrary to the hypothesis $f'(x)>0$.

Remark. The "first encounter argument" is also applicable to the proof of the fact that, for a convex function $y=f(x)$ with $f''(x)>0$, the graph $G(f;a,b)$ has no point which lies on the upper side of the line segment (the secant) connecting two points $\{a,f(a)\}$ and $\{b,f(b)\}$. We omit the details.

References

[1] L. Bers: Calculus. Holt, Rinehart and Winston Inc. (1969).
[2] P. Lax, S. Burstein, and A. Lax: Calculus with Applications and Computing, Volume I. Springer-Verlag (1976).