55. A Remark on the Completeness of the Bergman Metric

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§ 1. **Introduction.** The purpose of this note is to prove the following theorem by modifying the argument in a recent work of P. Pflug (cf. [4]).

Theorem 1. *Let D be a bounded pseudoconvex domain in C^n with a C^1-smooth boundary. Then D is complete with respect to*

$$d_D := \sum_{\alpha,\beta} \frac{\partial^2 \log K(z\,;D)}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha d\bar{z}^\beta.$$

Here we put $z=(z^1,\cdots,z^n)$ and denote by $K(z\,;D)$ the Bergman kernel function of D.

The metric d_D was first introduced by S. Bergman [1], and S. Kobayashi [2] asked "Which bounded domain (of holomorphy) in C^n is complete with respect to d_D?"

The author is grateful to Prof. P. Pflug who informed him of his very interesting result.

§ 2. **Preliminaries.** We put

$$K(z,w\,;D) := \sum_{i=1}^{\infty} f_i(z)\overline{f_i(w)},$$

where $\{f_i\}_{i=1}^\infty$ is an orthonormal basis of $L_h^2(D) := \{f\,;$ holomorphic, square integrable on $D\}$.

Lemma 1 (cf. Lemma 3 in [4, IV]). *Assume the sequence $\{z_\nu\}_{\nu=1}^\infty \subset D$ to be a Cauchy-sequence with respect to d_D. Then there exist a subsequence $\{z_{\nu(u)}\}_{u=1}^\infty$ and real numbers θ_u such that the sequence*

$$\left\{\frac{K(\ ,z_{\nu(u)}\,;D)}{K(z_{\nu(u)},z_{\nu(u)}\,;D)^{1/2}} e^{i\theta_u}\right\}_{u=1}^\infty$$

is a Cauchy-sequence in $L_h^2(D)$ whose members are all of modulus one.

From Lemma 1 we can deduce the following

Lemma 2. *Assume that d_D is not complete. Then there is a sequence $\{z_\nu\}_{\nu=1}^\infty \subset D$ converging to a point $z^* \in \partial D$, such that*

$$\lim_{\min\{\nu,\mu\}\to\infty} \left(1 - \frac{|K(z_\nu,z_\mu\,;D)|}{K(z_\nu,z_\nu\,;D)^{1/2}K(z_\mu,z_\mu\,;D)^{1/2}}\right) = 0.$$

Proof. We only have to note that

$$\left(\frac{K(\ ,z_\nu\,;D)}{K(z_\nu,z_\nu\,;D)^{1/2}}, \frac{K(\ ,z_\mu\,;D)}{K(z_\mu,z_\mu\,;D)^{1/2}}\right) = \frac{K(z_\mu,z_\nu\,;D)}{K(z_\nu,z_\nu\,;D)^{1/2}K(z_\mu,z_\mu\,;D)^{1/2}}.$$

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Let us consider the following condition:

(*) *$K(z, z; D)$ is exhausting and $H^\infty(D) := \{f$*; holomorphic, bounded on *$D\}$ is dense in $L^2_h(D)$.*

Theorem 2 (cf. the proof of Theorem in [4, IV]). *If D satisfies* (*), *then there is no sequence $\{z_\nu\}_{\nu=1}^\infty$ as in Lemma 2.*

We also need the following proposition which is a corollary to Satz 1 in [3].

Proposition 1. *If D is an intersection of domains with C^1-smooth boundaries, then $K(z, z; D)$ is exhausting.*

§3. Localization of the problem. We prove the following

Lemma 3. *Assume that for any point $z^* \in \partial D$, there is a neighbourhood U such that $U \cap D$ satisfies* (*). *Then d_D is complete.*

Proof. If d_D were not complete, then by Lemma 2 there would exist a sequence $\{z_\nu\}_{\nu=1}^\infty \subset D$ converging to a point $z^* \in \partial D$, such that

$$\lim_{\min\{\nu,\mu\}\to\infty}\left(1-\frac{|K(z_\nu, z_\mu; D)|}{K(z_\nu, z_\nu; D)^{1/2}K(z_\mu, z_\mu; D)^{1/2}}\right)=0.$$

By the definition of $K(z, z; D)$, this implies that for any positive number ε there exists an integer N such that, for any $\nu, \mu > N$, we can find no square integrable holomorphic function f satisfying

$$\begin{cases}\int_D |f|^2 dv = 1, \\ f(z_\nu)=0, \\ |f(z_\mu)| > \varepsilon \dfrac{|K(z_\mu, z_\nu; D)|}{K(z_\nu, z_\nu; D)^{1/2}},\end{cases}$$

where dv denotes the Lebesgue measure. On the other hand, there exists a neighbourhood U of z^* such that $U \cap D$ satisfies (*). Thus, by Theorem 2, there exists a positive number δ such that, for any choice of the above ε and N, we can find $\nu, \mu > N$ such that

$$\frac{|K(z_\mu, z_\nu; U\cap D)|}{K(z_\nu, z_\nu; U\cap D)^{1/2}K(z_\mu, z_\mu; U\cap D)^{1/2}} > 1-\delta.$$

We put

$$a_{\nu\mu}=\frac{K(z_\nu, z_\mu; U\cap D)}{K(z_\nu, z_\nu; U\cap D)^{1/2}K(z_\mu, z_\mu; U\cap D)^{1/2}}$$

and

$$f_\nu=\frac{K(\ , z_\nu; U\cap D)}{K(z_\nu, z_\nu; U\cap D)^{1/2}}.$$

Then we have

$$\begin{cases} f_\mu(z_\nu) - a_{\nu\mu} f_\nu(z_\nu) = 0 \\ |f_\mu(z_\mu) - a_{\nu\mu} f_\nu(z_\mu)| \geqq (1-|a_{\nu\mu}|^2)|f_\mu(z_\mu)|. \end{cases}$$

By a standard method*) we can find a square integrable holomorphic function $h_{\nu\mu}$ on D such that

*) L^2 estimate of $\bar\partial$ with weight $\exp(-n\log\sum_{\alpha=1}^n |z^\alpha - z_\nu^\alpha|^2 - n\log\sum_{\alpha=1}^n |z^\alpha - z_\mu^\alpha|^2)$.

$$\begin{cases} \int_D |h_{\nu\mu}|^2 dv \leqq K \int_D |f_\mu - a_{\nu\mu} f_\nu|^2 dv, \\ h_{\nu\mu}(z_\nu) = 0, \\ h_{\nu\mu}(z_\mu) = f_\mu(z_\mu) - a_{\nu\mu} f_\nu(z_\mu), \end{cases}$$

where K is a constant which does not depend on ν nor μ. Thus, choosing ε sufficiently small, we get a contradiction.

§ 4. **Proof of Theorem 1.** In virtue of Lemma 3, we only have to find, for any point $z^* \in \partial D$, a neighbourhood U such that $U \cap D$ satisfies the condition $(*)$. Since the boundary of D is C^1-smooth, we can take a ball B in D such that $\bar{B} \cap \partial D = \{z^*\}$. To obtain a required neighbourhood U of z, we only need to slide B slightly in the direction of the outer normal of ∂D at z^*. That $K(z, z; U \cap D)$ is exhausting follows from Proposition 1, and that $H^\infty(U \cap D)$ is dense in $L_h^2(U \cap D)$ follows from the existence of the homothetic transformation $f(z) \to f(az)$ $(|a| < 1)$ on $L_h^2(U \cap D)$, where the center of U is taken to be the origin. This completes the proof of Theorem 1.

References

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