48. *The Application of Monodromy Preserving Deformation to the Gravitational Field Equation*

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§ 0. In this note, we will show the new method for constructing exact solutions of the vacuum Einstein equation for stationary axi-symmetric gravitational fields (VESA).

From a viewpoint of the inverse scattering theory, Belinsky-Zakharov (B-Z) [1], [2] gave an interesting method for integrating VESA, expressed by the metric form

$$-ds^2=f(d\rho^2+dz^2)+g_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha, \beta=0, 1) \tag{0.1}$$

where f and $g_{\alpha\beta}$ are functions in ρ and z, and x^0, x^1 represent the coordinates t, ϕ, respectively.

Under the supplementary condition

$$\det g=-\rho^2, \qquad g=(g_{\alpha\beta}), \tag{0.2}$$

the fields equation for the metric (0.1) can be written as follows:

$$\begin{cases} U_\rho+V_z=0 \\ U_z-V_\rho+\rho^{-1}V+\rho^{-1}[U, V]=0 \end{cases} \tag{0.3}$$

$$\begin{cases} (\log f)_\rho=-\rho^{-1}+(4\rho)^{-1}\operatorname{trace}(U^2-V^2) \\ (\log f)_z=(2\rho)^{-1}\operatorname{trace}(UV). \end{cases} \tag{0.4}$$

Here $U=\rho g_\rho g^{-1}$, and $V=\rho g_z g^{-1}$. We should note that the matrix g is symmetric. B-Z found that the equation (0.3) are equivalent to the compatibility conditions of the system of linear equations

$$\begin{cases} D_1 Y=\dfrac{\rho V-\lambda U}{\lambda^2+\rho^2}Y, \\ D_2 Y=\dfrac{\lambda V+\rho U}{\lambda^2+\rho^2}Y, \end{cases} \tag{0.5}$$

where

$$D_1=\frac{\partial}{\partial z}-\frac{2\lambda^2}{\lambda^2+\rho^2}\frac{\partial}{\partial\lambda}, \qquad D_2=\frac{\partial}{\partial\rho}+\frac{2\lambda\rho}{\lambda^2+\rho^2}\frac{\partial}{\partial\lambda},$$

and λ is a complex parameter independent of ρ and z.

If we find a solution $Y(\lambda)=Y(\lambda, \rho, z)$ to (0.4), and set

$$g=Y(0)=Y(0, \rho, z), \tag{0.6}$$

the potentials U and V in (0.5) can be recovered as $U=\rho g_\rho g^{-1}$, $V=\rho g_z g^{-1}$, so we obtain a solution of (0.3). But we should note that the function g given by (0.6) is not always assured to be symmetric, real, and to satisfy the condition (0.2). We can easily find the conditions that g is real and satisfies (0.2) (cf. [1], [2], [9]). Therefore one of the crucial points to which we must make efforts is to find out the symmetric conditions for $g=Y(0)$.

By applying monodromy preserving deformation (MPD), we are succeed in construction of exact solutions of VESA. Namely, our main tool here is MPD of a certain Fuchsian equation

$$\frac{dY}{d\lambda}=\sum_{j=1}^{n}\frac{A_j}{\lambda-\mu_j}Y. \tag{0.7}$$

If the global monodromy of the normalized fundamental solution matrix (briefly, the normalized solution) $Y(\lambda)=Y(\lambda, \rho, z)$ is kept, and at the same time, satisfy appropriate conditions, then Y solves the equation (0.3) with some potential U, V, and its 0-value $Y(0)=Y(0, \rho, z)$ is symmetric.

This is our main result in this article.

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§1. MPD and VESA. In order to construct special solutions of (0.3), instead of (0.5), we start from a 2×2 Fuchsian equation in the complex domain

$$\frac{dY}{d\lambda}=\sum_{j=1}^{m}\frac{A_j}{\lambda-\mu_j}Y \tag{1.1}$$

where each singular point μ_j is a root of the quadratic equation

$$\mu^2-2(w_j-z)\mu-\rho^2=0, \qquad w_j\in \boldsymbol{C}, \tag{1.2}$$

namely μ_j is a solution of non-linear equations of first order

$$\frac{\partial\mu}{\partial z}=\frac{-2\mu^2}{\mu^2+\rho^2}, \qquad \frac{\partial\mu}{\partial\rho}=\frac{2\rho\mu}{\mu^2+\rho^2}. \tag{1.3}$$

We assume that, in this equation, the exponents at each singular point are distinct modulo integers, and that there exists the solution normalized at the infinity $Y^{(\infty)}(\lambda)=Y^{(\infty)}(\lambda, \rho, z)$, which has a local expansion at the infinity

$$Y^{(\infty)}=(1+\hat{Y}_1^{(\infty)}\lambda^{-1}+\cdots)(\lambda^{-1})^{L^{(\infty)}}, \qquad \text{as } \lambda\to\infty, \tag{1.4}$$

where $L^{(\infty)}$ is a diagonal matrix.

First we obtain the following fundamental theorem.

Theorem 1.1. *If the global monodromy of $Y^{(\infty)}(\lambda)$ is independent of ρ and z* (i.e. isomonodromic), *$Y^{(\infty)}(\lambda)$ solves the equation of* (0.5), *where the potentials U and V are given by*

$$U=\sum_{j=1}^{n}\frac{2\rho^2A_j}{\mu_j^2+\rho^2}, \qquad V=-\sum_{j=1}^{n}\frac{2\rho\mu_jA_j}{\mu_j^2+\rho^2}. \tag{1.5}$$

This theorem is very important, and plays a central role in our scheme. However we should note that the symmetry of $Y^{(\infty)}(0)$ is not always assured. In order to describe the symmetric condition for $Y^{(\infty)}(0)$ as that of the global monodromy, we need the following proposition, which was suggested by Prof. E. Date (cf. [3]).

Proposition 1.2 (Date). *Let $Y(\lambda)=Y(\lambda,\rho,z)$ be a solution of* (0.5). *If $Y(0)$ is symmetric, there exists an invertible matrix $S(\lambda,\rho,z)$ such that*

$$D_1S=D_2S=0, \tag{1.9}$$

and

$$Y(0)=Y\left(-\frac{\rho^2}{\lambda}\right)S(\lambda)\,{}^tY(\lambda). \tag{1.10}$$

Here ${}^tY(\lambda)$ denotes the transposed matrix of $Y(\lambda)$.

For the purpose of the interpretation to the feature of $S(\lambda)$, we introduce a variable w through

$$w=2^{-1}(\lambda+2z-\lambda^{-1}\rho^2). \tag{1.11}$$

$S(\lambda)$ is a function of w, and if μ is a root of (1.2), $S(\mu)$ is constant. $S(\lambda)$ has also a symmetric property

$$S(\lambda)={}^tS\left(-\frac{\rho^2}{\lambda}\right). \tag{1.12}$$

Let $Y_0(\lambda)$ be a solution of (0.5) with potentials U_0 and V_0. Suppose that $Y_0(0)$ is symmetric. We define a function $S_0(\lambda)$ by (1.10), where $Y(\lambda)$ is replaced by $Y_0(\lambda)$. Furthermore, let $Y(\lambda)$ be a solution of (0.5) with new potentials U and V. Then we consider the condition that $Y(-(\rho^2/\lambda))S_0(\lambda)\,{}^tY(\lambda)$ just gives $Y(0)$.

Proposition 1.3. *Suppose that $g=Y(-(\rho^2/\lambda))S_0(\lambda)\,{}^tY(\lambda)$ is independent of λ. Then the 0-value of $Y(\lambda)$ is given by*

$$Y(0)=g, \tag{1.13}$$

moreover $Y(0)$ is symmetric.

According to this proposition, we search for the condition that the normalized solution $Y^{(\infty)}(\lambda)$ of a 2×2 Fuchsian equation

$$\frac{dY}{d\lambda}=\sum_{j=1}^{n}\left(\frac{A_j}{\lambda-a_j}+\frac{B_j}{\lambda-b_j}\right)Y \tag{1.14}$$

has the symmetric 0-value $Y^{(\infty)}(0)$. In (1.14), a_j, b_j are two roots of a quadratic equation of (1.2). We fix a function $y_0(\lambda)$,

$$y_0(\lambda)=\prod_{j=1}^{n}(\lambda-a_j)^{\alpha_j}(\lambda-b_j)^{\beta_j}, \qquad \alpha_j,\ \beta_j\in \boldsymbol{C}, \tag{1.15}$$

associated with (1.14), and set

$$S_0(\lambda)=y_0\left(-\frac{\rho^2}{\lambda}\right)^{-1}y_0(0)y_0(\lambda)^{-1}. \tag{1.16}$$

The normalized solution $Y^{(\infty)}(\lambda)$ is assumed to have a local expansion at each singular point

$$y_0(\lambda)Y^{(\infty)}(\lambda)^{-1}|_{\lambda=\infty}=1, \tag{1.17}$$

$$Y^{(\infty)}(\lambda)=G^{(\mu)}\hat{Y}^{(\mu)}(\lambda)(\lambda-\mu)^{L^{(\mu)}}C^{(\mu)}, \tag{1.18}$$

$$\hat{Y}^{(\mu)}(\lambda)=\sum_{l=1}^{\infty}Y_l^{(\mu)}(\lambda-\mu)^l,\quad Y_0^{(\mu)}=1,\quad \mu=a_j,\ b_j \quad (j=1,\cdots,n),$$

(1.19) $\quad L^{(\mu)}=\operatorname{diag}(l_1^{(\mu)}, l_2^{(\mu)})$ and $C^{(\mu)}$ is constant, invertible.
Namely the global monodromy of $Y^{(\infty)}(\lambda)$ is preserved.

We state our main result.

Theorem 1.4. *Suppose that the entries of any $L^{(\mu)}$ are distinct modulo integers. Then, $g=Y^{(\infty)}(-(\rho^2/\lambda))S_0(\lambda)^t Y^{(\infty)}(\lambda)$ is independent of λ, if and only if either of the following conditions is always valid for any j:*

(1.20) $$l_s^{(b_j)}+l_s^{(a_j)}=\beta_j+\alpha_j, \qquad \textit{for } s=1, 2,$$

and $C^{(b_j,a_j)}$ is a diagonal matrix,

(1.21) $$l_s^{(b_j)}+l_{s'}^{(a_j)}=\beta_j+\alpha_j, \quad \textit{for } s\neq s', \quad s, s'=1, 2$$

and $C^{(b_j,a_j)}$ is a diagonal free matrix, where $C^{(b_j,a_j)}=C^{(b_j)\,t}C^{(a_j)}$.

For the proof, the reader should be referred to [9].

Using this theorem, together with Theorem 1.1 and Proposition 1.3, we can find out the symmetric solution g of (0.3).

§ 2. **Schlesinger transformations and symmetry.** In this section, we will discuss Schlesinger transformations which preserve the symmetry of $Y^{(\infty)}(0)$.

We make a Schlesinger transformation of type

(2.1) $$\begin{Bmatrix} a_{n+1}, \cdots, a_{n+N} & b_{n+1}, \cdots, b_{n+N} \\ -E_1 \cdots, -E_1 & E_1, \cdots, E_1 \end{Bmatrix},$$
$$E_1=\operatorname{diag}(1, 0), \quad E_2=\operatorname{diag}(0, 1),$$

to the normalized solution $Y^{(\infty)}(\lambda)$ of (1.1) (cf. [8]). Here a_j and b_j $(j=n+1, \cdots, n+N)$ are two roots of the quadratic equation

(2.2) $$\mu^2-2(w_j-z)\mu-\rho^2=0, \quad w_j\in \boldsymbol{C}, \quad w_j\neq w_k, \quad (j\neq k)$$
$$j=n+1, \cdots, n+N,$$

and assumed to be regular points of (1.1). At these regular points, a gauge matrix $G^{(\mu)}$ and a connection matrix $C^{(\mu)}$ $(\mu=a_j, b_j)$ are introduced through

(2.3) $\quad C^{(\mu)}$; an arbitrary constant invertible matrix,
$$G^{(\mu)}=Y^{(\infty)}(\mu)C^{(\mu)-1}.$$

The multiplier $R(\lambda)$ for the transformation (2.1) is given by

(2.4) $$R(\lambda)=1+\sum_{j=n+1}^{n+N}\frac{R_j}{\lambda-a_j}$$
$$=1+[\vec{G}^{(b_{n+1})}, \cdots, \vec{G}^{(b_{n+N})}]W^{-1}\begin{bmatrix} \dfrac{1}{\lambda-a_{n+1}}\,{}^t\vec{G}^{(a_{n+1})-1} \\ \vdots \\ \dfrac{1}{\lambda-a_{n+N}}\,{}^t\vec{G}^{(a_{n+N})-1} \end{bmatrix},$$

(2.5) $$W=\left[\frac{(G^{(a_{n+i})-1}G^{(b_{n+j})})_{11}}{a_{n+i}-b_{n+j}}\right]_{i,j=1,\cdots,N},$$

(2.6) $${}^t\vec{G}^{(b_{n+j})}=[G_{11}^{(b_{n+j})}, G_{21}^{(b_{n+j})}],$$
$${}^t\vec{G}^{(a_{n+j})-1}=[G_{11}^{(a_{n+j})-1}, G_{12}^{(a_{n+j})-1}].$$

Throughout the following discussion, $Y^{(\infty)}(0)$ is assumed to be symmetric. We define $S(\lambda)$ by (1.10), replacing $Y(\lambda)$ by $Y^{(\infty)}(\lambda)$, and set

(2.7) $$Y_1^{(\infty)}(\lambda)=R(\lambda)Y^{(\infty)}(\lambda).$$

Then $Y_1^{(\infty)}(\lambda)$ is a solution normalized at the infinity of a certain Fuchsian equation with singular points as $\mu_1, \cdots, \mu_m, a_{n+1}, \cdots, b_{n+N}$, and if the global monodromy of $Y_1^{(\infty)}(\lambda)$ is kept, that of $Y_1^{(\infty)}(\lambda)$ is so. Therefore $Y_1^{(\infty)}(\lambda)$ solves the equation (0.5), whose potentials are replaced by, say, U_1 and V_1. But we should note that $Y_1^{(\infty)}(0)$ is not always symmetric even if $Y^{(\infty)}(0)$ is so. Along the scheme stated in § 1, we consider $g_1=Y_1^{(\infty)}(-(\rho^2/\lambda))S(\lambda)^tY_1^{(\infty)}(\lambda)$. By Proposition 1.3, if g_1 does not depend on λ, $g_1=Y_1(0)$, and g_1 is symmetric. In a similar way as Theorem 1.4, we obtain the following

Theorem 2.1. *$g_1=Y_1^{(\infty)}(-(\rho^2/\lambda))S(\lambda)^tY_1^{(\infty)}(\lambda)$ does not depend on λ, hence $g_1=Y_1^{(\infty)}(0)$ is symmetric, if and only if the following condition holds for each $j=n+1, \cdots, n+N$.*

(2.8) $$E_2C^{(b_j)}S(a_j)^tC^{(a_j)}E_1=0.$$

Remark. The Schlesinger transformations considered in Theorem 2.1 are equal to those constructed in [1], [2], because the following fact holds:

"$g_1=Y_1^{(\infty)}(-(\rho^2/\lambda))S(\lambda)^tY_1^{(\infty)}(\lambda)$ is independent of λ" is equivalent to "$g_1=R(-(\rho^2/\lambda))Y^{(\infty)}(0)^tR(\lambda)$ is independent of λ".

In [1], [2], B-Z determines the coefficients of $R(\lambda)$, by the latter condition.

§ 3. The τ-function and the metric coefficient f. In this section, we will show that the τ-function for the Schlesinger equation is essentially the metric coefficient f, defined by (0.4). Throughout this section, the equation (1.1) and its normalized solution $Y^{(\infty)}(\lambda)$ satisfy the condition in the situation of Theorem 1.1.

First we interpret the τ-function for the Schlesinger equation (cf. [4], [7]). The Schlesinger equation is a deformation equation for Fuchsian system (1.1), and a completely integrable system, given by

(3.1) $$dA_j=\sum_{k(\neq j)}[A_k, A_j]\frac{d(\mu_k-\mu_j)}{\mu_k-\mu_j}, \qquad j=1, \cdots, m.$$

The τ-function for (3.1) is defined by

(3.2) $$d\log\tau=\frac{1}{2}\sum_{j\neq k}\operatorname{trace} A_jA_k\frac{d(\mu_j-\mu_k)}{\mu_j-\mu_k},$$

which is a closed 1-form under (3.1). The metric coefficient f is introduced through (0.4), where the potentials U and V are given by (1.5).

By a direct computation, we can prove our first assertion.

Theorem 3.1. *Under the condition of Theorem* 1.1, *we have*

(3.3) $$\rho f=\text{const}\prod_{j=1}^{m}\left(\frac{\mu_j^2}{\mu_j^2+\rho^2}\right)^{1/2\operatorname{trace} L^{(\mu_j)2}}\tau,$$

where $L^{(\mu_j)}$ is an exponent matrix of the monodromy of $Y^{(\infty)}(\lambda)$ around μ_j.

Next we consider how f changes under the Schlesinger transformation of type (2.1). Let $Y_1^{(\infty)}(\lambda)$ be given by (5.1), and f, f_1 the metric coefficient corresponding to $Y^{(\infty)}(\lambda)$, $Y_1^{(\infty)}(\lambda)$, respectively. By Theorem 3.1, together with Theorem 4.1 in Jimbo-Miwa [8], we obtain the following

Theorem 3.2. *Let W be an $N\times N$ matrix given by* (2.5). *Then we have*

$$f_1=\text{const}\, f\rho^N \prod_{j=1}^{N}\left(\frac{a_{n+j}}{a_{n+j}^2+\rho^2}\right)\det W. \tag{3.4}$$

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