45. *Hopf Bifurcation of Semilinear Evolution Equations*

By Tatsuo ITOH
Department of Pure and Applied Sciences, College of General Education, University of Tokyo

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1. Introduction and the assumptions. The present paper is concerned with the two problems. The first problem is the Hopf bifurcation problem for a semilinear evolution equation in a real Banach space X (with norm $\| \ \|$) with a real parameter λ;

(E) $$du/dt = Lu + N(u, \lambda) \qquad t>0.$$

The second one is to determine a local ω-limit set of a solution $u(t, x_0)$ of a semilinear evolution equation in X;

(E′) $$du/dt = Lu + N(u) \qquad t>0$$

with an initial value: $u(0)=x_0$. Here we assume

Assumption 1. L is the generator of the holomorphic semigroup, having $\pm i$ as isolated eigenvalues with the algebraic multiplicity one and the other spectrum $\sigma'(L)$ of L being properly contained in the left half-(complex)plane;

$$\sup_{\mu \in \sigma'(L)} \operatorname{Re} \mu < -c$$

(c: a positive constant).

Assumption 2. $N(x, \lambda)$ is a C^3-mapping of a neighborhood of 0 in $X \times R^1$ into X such that $N(0, 0)=0$, $D_x N(0, 0)=0$. ($D_x N(0, 0)$ means the Fréchet derivative of $N(x, \lambda)$ with respect to x at $x=0$, $\lambda=0$.)

Assumption 2′. $N(x)$ is a C^3-mapping of a neighborhood V of 0 (in X) into X such that $N(0)=0$.

Before stating our results, we shall give the definition of a local ω-limit set of a solution $u(t, x_0)$ of (E'). Let U_1, U_2 be neighborhoods of 0 with $U_1 \subset U_2 \subset V$. For $x_0 \in U_1$ we define a local ω-limit set $\Omega_{U_1, U_2}(x_0)$ of a solution $u(t, x_0)$ of (E') by

$$\Omega_{U_1,U_2}(x_0) = \begin{cases} \bigcap_{s \geq 0} \text{closure } \{u(t, x_0)\,;\ t \geq s\} & (\text{if } u(t, x_0) \in U_2,\ t \geq 0) \\ \phi & (\text{otherwise}) \end{cases}.$$

2. Results. Theorem 1. *Under Assumptions* 1 *and* 2, *if a null solution* 0 *of* (E) *changes its stability at* $\lambda=0$, *then non-stationary periodic orbits bifurcate from* $(x, \lambda)=(0, 0)$.

Theorem 2. *Under Assumptions* 1 *and* 2′, *there exists a neighborhood* $U_1(\subset V)$ *of* 0 *such that if* $\sup_{x \in V} \|D_x N(x)\|$ *is sufficiently small, then for some* $U_2(U_1 \subset U_2 \subset V)$ *and for any* $x_0 \in U_1$ *with* $\Omega_{U_1,U_2}(x_0) \neq \phi$, $\Omega_{U_1,U_2}(x_0)$ *consists only of a periodic orbit* $\gamma(x_0)$ *of* (E′) *in* U_2 ($\gamma(x_0)$ *may be* $\{0\}$).

Moreover, $u(t, x_0) \to \gamma(x_0)$ $(t \to \infty)$. *In particular, the condition* $\Omega_{U_1,U_2}(x_0) \neq \phi$ *is satisfied if we further assume*; (i) *a null solution* 0 *is unstable, and* (ii) *there exists a unique non-stationary periodic orbit of* (E′) *in* U_2, *which is stable.*

The proofs of theorems will be published elsewhere.

3. Remarks. Remark 1. Let $\kappa(\lambda)$ be the eigenvalue near i of the linearized operator $L + D_x N(0, \lambda)$. E. Hopf [3] showed that if

$$\partial \operatorname{Re} \kappa(0)/\partial\lambda \neq 0, \tag{1}$$

then non-stationary periodic orbits of (E) bifurcate from $(x, \lambda) = (0, 0)$. The condition (1) can be replaced by the condition (2): $\operatorname{Re} \kappa(\lambda) > 0$ $(\lambda > 0)$ and a null solution 0 is asymptotically stable at $\lambda = 0$ ([1]), or the condition (3): $\operatorname{Re} \kappa(\lambda)$ changes its sign at $\lambda = 0$ ([4]). All the above conditions are sufficient in order that a null solution 0 changes its stability at $\lambda = 0$ ([2]).

Remark 2. Under the condition (2), Chafee [1] showed that if $\lambda > 0$ is sufficiently small and if an initial value x_0 is sufficiently near 0, then as $t \to \infty$, a solution $u(t, x_0, \lambda)$ of (E) converges either to $\{0\}$ or to an invariant set, which lies on a locally invariant manifold of dimension two.

References

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