44. *Another Construction of Lie Algebras by Generalized Jordan Triple Systems of Second Order*

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Introduction. U. Hirzebruch [3] and G. Rhinow [9] have generalized Tits' construction of Lie algebras by Jordan algebras [11] to Jordan triple systems (JTS), using a certain two dimensional JTS. Moreover H. Asano and K. Yamaguti [2] have generalized Hirzebruch's construction to generalized JTS of second order (due to I. L. Kantor [4]), using the same two dimensional JTS. In this note, it is shown that Lie algebras can be also constructed by generalized JTS of second order (gen. JTS of 2nd order), using a certain two dimensional associative triple system (ATS) (cf. [6]). From a two dimensional triple system W and any gen. JTS $\mathfrak{J}$ of 2nd order, we make a gen. JTS $W\otimes\mathfrak{J}$ of 2nd order, where W is a certain ATS (see § 1) while in [2], W was a certain JTS. In both cases, Lie algebras can be constructed from $W\otimes\mathfrak{J}$. In other words, Lie algebras can be constructed from gen. JTS $(\mathfrak{J}\oplus\mathfrak{J})_\varepsilon$ of 2nd order (see § 2) where in case $\varepsilon=-1$ we have the Asano–Yamaguti construction and in case $\varepsilon=+1$, we obtain our construction in this note. We assume that any vector space considered in this note is finite dimensional and the characteristic of base field Φ is different from 2 or 3. The author wishes to express his hearty thanks to Prof. K. Yamaguti for his kind advices and encouragements.

§ 1. A triple system satisfying $\{ab\{cde\}\}=\{a\{bcd\}e\}=\{\{abc\}de\}=\{a\{dcb\}e\}$ for any elements a, b, c, d, e is called an ATS.

Let W be a two dimensional triple system which has a basis $\{e_1, e_2\}$ such that

$$(1)\qquad \begin{aligned}&\{e_1e_1e_1\}=\alpha e_1, \quad \{e_1e_1e_2\}=\{e_1e_2e_1\}=\{e_2e_1e_1\}=\alpha e_2,\\ &\{e_1e_2e_2\}=\{e_2e_1e_2\}=\{e_2e_2e_1\}=\beta e_1, \quad \{e_2e_2e_2\}=\beta e_2,\end{aligned}$$

where $\alpha, \beta\in\Phi$. Then W is a commutative ATS and is also a JTS. In the ATS W, we have

$$(2)\qquad l(a, b)l(c, d)=l(c, d)l(a, b),$$

$$(3)\qquad l(a, b)l(c, d)=l(l(a, b)c, d)=l(c, l(b, a)d),$$

where $l(a, b)c=\{abc\}$, for $a, b, c, d\in W$.

A gen. JTS $\mathfrak{J}$ of 2nd order is a vector space with a triple product $\{xyz\}$ satisfying

(4) $[L(x, y), L(u, v)] = L(L(x, y)u, v) - L(u, L(y, x)v),$

(5) $K(K(x, y)u, v) - L(v, u)K(x, y) - K(x, y)L(u, v) = 0,$

where $L(x, y)u = \{xyu\}$ and $K(x, y)u = \{xuy\} - \{yux\}$, for $x, y, u, v \in \mathfrak{J}$ ([4]).

Hence a JTS is a gen. JTS of 2nd order such that K vanishes identically.

Using (2) and (3), we have

Lemma 1. *For the* ATS W *and any gen.* JTS $\mathfrak{J}$ *of* 2*nd order, define a trilinear product in* $W \otimes \mathfrak{J}$ *by* $\{a \otimes x\ b \otimes y\ c \otimes z\} = \{abc\} \otimes \{xyz\}$ *for* $a, b, c \in W$, $x, y, z \in \mathfrak{J}$. *Then* $W \otimes \mathfrak{J}$ *becomes a gen.* JTS *of* 2*nd order.*

A triple system is called a Lie triple system (LTS) if it satisfies the following identities for any elements x, y, z, u, v ([5]):

(i) $[xxy] = 0,$

(ii) $[xyz] + [yzx] + [zxy] = 0,$

(iii) $[xy[uvz]] = [[xyu]vz] + [u[xyv]z] + [uv[xyz]].$

Let $\mathfrak{J}$ be a gen. JTS of 2nd order with product $\{xyz\}$. It is known ([2]) that $\mathfrak{J}$ becomes a LTS relative to a new product $[xyz] := \{xyz\} - \{yxz\} + \{xzy\} - \{yzx\}$. We denote this LTS by $\mathfrak{J}^*$ and call this a LTS induced by $\mathfrak{J}$ or an induced LTS (from $\mathfrak{J}$). For the gen. JTS $W \otimes \mathfrak{J}$ of 2nd order in Lemma 1, the Lie triple product (LT product) in $(W \otimes \mathfrak{J})^*$ is as follows: $[a \otimes x\ b \otimes y\ c \otimes z] = \{abc\} \otimes [xyz]$. Hence $D(a \otimes x, b \otimes y) = l(a, b) \otimes D(x, y)$ where $D(x, y)z := [xyz]$ and $D(a \otimes x, b \otimes y)(c \otimes z) := [a \otimes x\ b \otimes y\ c \otimes z]$. Let $\mathfrak{D}$ be the Lie algebra of inner derivations in the LTS $(W \otimes \mathfrak{J})^*$, then $\mathfrak{S}(W, \mathfrak{J}) = \mathfrak{D} \oplus (W \otimes \mathfrak{J})^*$ is the standard enveloping Lie algebra of the LTS $(W \otimes \mathfrak{J})^*$. If $\alpha \neq 0$ or $\beta \neq 0$ in W, then $\{id_W, l(e_1, e_2)\}$ is a basis of the vector space $l(W, W)$ spanned by $\{l(a, b) : a, b \in W\}$, where id_W is the identity endomorphism in the ATS W. Hence $\mathfrak{D} = id_W \otimes D(\mathfrak{J}, \mathfrak{J}) \oplus l(e_1, e_2) \otimes D(\mathfrak{J}, \mathfrak{J})$, where $D(\mathfrak{J}, \mathfrak{J})$ is the Lie algebra of inner derivations in $\mathfrak{J}^*$. Then we have the following

Theorem 1. *If* $\alpha \neq 0$ *or* $\beta \neq 0$ *in the* ATS W, *then*

$$id_W \otimes D(\mathfrak{J}, \mathfrak{J}) \oplus l(e_1, e_2) \otimes D(\mathfrak{J}, \mathfrak{J}) \oplus (W \otimes \mathfrak{J})^*$$

is the standard enveloping Lie algebra of the LTS $(W \otimes \mathfrak{J})^*$. *And,* $id_W \otimes D(\mathfrak{J}, \mathfrak{J}) \oplus l(e_1, e_2) \otimes D(\mathfrak{J}, \mathfrak{J})$ *is a Lie subalgebra satisfying the following commutator relations:*

$$[\mathfrak{L}, \mathfrak{L}] \subset \mathfrak{L}, \quad [\mathfrak{L}, \mathfrak{M}] \subset \mathfrak{M}, \quad [\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{L},$$

where $\mathfrak{L} := id_W \otimes D(\mathfrak{J}, \mathfrak{J})$ *and* $\mathfrak{M} := l(e_1, e_2) \otimes D(\mathfrak{J}, \mathfrak{J})$.

§2. Let $\mathfrak{J}$ be a gen. JTS of 2nd order. Now we consider the vector space direct sum $\mathfrak{J} \oplus \mathfrak{J}$, of which element is denoted by $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and define a triple product on it by

$$(6)\ \left\{\begin{pmatrix}x_1\\x_2\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}\right\}:=\begin{pmatrix}\alpha\{x_1\,y_1\,z_1\}+\beta\{x_1\,y_2\,z_2\}+\varepsilon\beta\{x_2\,y_1\,z_2\}+\beta\{x_2\,y_2\,z_1\}\\ \alpha\{x_1\,y_1\,z_2\}+\varepsilon\alpha\{x_1\,y_2\,z_1\}+\alpha\{x_2\,y_1\,z_1\}+\beta\{x_2\,y_2\,z_2\}\end{pmatrix}$$
$$=\begin{pmatrix}\alpha L(x_1, y_1)+\beta L(x_2, y_2) & \beta L(x_1, y_2)+\varepsilon\beta L(x_2, y_1)\\ \varepsilon\alpha L(x_1, y_2)+\alpha L(x_2, y_1) & \alpha L(x_1, y_1)+\beta L(x_2, y_2)\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix},$$

where α, β, ε $(=\pm 1)$ are the elements of the base field Φ.

By straightforward calculations we have

Theorem 2. *Let $\mathfrak{J}$ be a gen.* JTS *of 2nd order, then $\mathfrak{J}\oplus\mathfrak{J}$ is a gen.* JTS *of 2nd order relative to the product defined above.*

The gen. JTS of 2nd order obtained in Theorem 2 is denoted by $(\mathfrak{J}\oplus\mathfrak{J})_\varepsilon$. For $\varepsilon=1$, if we define a linear mapping f of $W\otimes\mathfrak{J}$ into $(\mathfrak{J}\oplus\mathfrak{J})_{+1}$ by $f(e_1\otimes x_1+e_2\otimes x_2)=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, we have the following

Theorem 3. *$W\otimes\mathfrak{J}$ is isomorphic to $(\mathfrak{J}\oplus\mathfrak{J})_{+1}$ as gen.* JTS *of 2nd order.*

By direct calculations, we see that the product in the induced LTS $(\mathfrak{J}\oplus\mathfrak{J})^*_\varepsilon$ is given as follows

$$(7)\quad \left[\begin{pmatrix}x_1\\x_2\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}\right]=\begin{pmatrix}\alpha[x_1\,y_1\,z_1]+\beta[x_1\,y_2\,z_2]+\varepsilon\beta[x_2\,y_1\,z_2]+\beta[x_2\,y_2\,z_1]\\ \alpha[x_1\,y_1\,z_2]+\varepsilon\alpha[x_1\,y_2\,z_1]+\alpha[x_2\,y_1\,z_1]+\beta[x_2\,y_2\,z_2]\end{pmatrix},$$

where $[xyz]$ is the product in the LTS $\mathfrak{J}^*$.

Remark 1. If we put $\varepsilon=-1$ in (6), $(\mathfrak{J}\oplus\mathfrak{J})_{-1}$ is isomorphic to $J(\alpha, \beta, 0)$ in [2]. Hence Lie algebras can be constructed by $(\mathfrak{J}\oplus\mathfrak{J})_{-1}$ as in [2].

For an induced LTS $\mathfrak{J}^*$, we consider the vector space direct sum $\mathfrak{J}^*\oplus\mathfrak{J}^*$, of which element is denoted by $\begin{pmatrix}x_1\\x_2\end{pmatrix}$. Then, using the expression (7) we obtain

Theorem 4. *If in $\mathfrak{J}^*\oplus\mathfrak{J}^*$ we define a triple product by*

$$(8)\quad \left[\begin{pmatrix}x_1\\x_2\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}\right]=\begin{pmatrix}\alpha[x_1\,y_1\,z_1]+\beta[x_1\,y_2\,z_2]+\beta[x_2\,y_1\,z_2]+\beta[x_2\,y_2\,z_1]\\ \alpha[x_1\,y_1\,z_2]+\alpha[x_1\,y_2\,z_1]+\alpha[x_2\,y_1\,z_1]+\beta[x_2\,y_2\,z_2]\end{pmatrix},$$

then $\mathfrak{J}^*\oplus\mathfrak{J}^*$ becomes a LTS *and is isomorphic to $(\mathfrak{J}\oplus\mathfrak{J})^*_{+1}$ as* LTS.

Remark 2. If we put $\alpha=1$ and $\beta=0$, ± 1 in the product (8), we get the LT product defined by Y. Taniguchi (cf. [10]).

§ 3. K. Yamaguti has defined a bilinear form $\gamma_{\mathfrak{J}}$ of a gen. JTS $\mathfrak{J}$ of 2nd order by

$$\gamma_{\mathfrak{J}}(x, y)=\frac{1}{2}Sp[2(R(x, y)+R(y, x))-L(x, y)-L(y, x)],$$

where $R(x, y)z=\{zxy\}$ ([12]). Using this definition, the bilinear forms γ_W, γ_1 and γ_2 of W, $W\otimes\mathfrak{J}$ and $(\mathfrak{J}\oplus\mathfrak{J})_\varepsilon$ are as follows: $\gamma_W(a, b)=Sp\ l(a, b)$, $\gamma_1(a\otimes x, b\otimes y)=\gamma_W(a, b)\gamma_{\mathfrak{J}}(x, y)$ and $\gamma_2\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}\right)=2\alpha\gamma_{\mathfrak{J}}(x_1, y_1)+2\beta\gamma_{\mathfrak{J}}(x_2, y_2)$ respectively.

The Killing form κ of LTS $\mathfrak{J}^*$ is given as: $\kappa(x, y)=(1/2)Sp[R(x, y)+R(y, x)]$, where $R(x, y)z=[zxy]$ ([8]). Then, the Killing forms κ_1 and

κ_2 of $(W\otimes\mathfrak{J})^*$ and $(\mathfrak{J}\oplus\mathfrak{J})^*_\epsilon$ are as follows: $\kappa_1(a\otimes x, b\otimes y)=\gamma_W(a, b)\kappa(x, y)$ and $\kappa_2\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}\right)=2\alpha\kappa(x_1, y_1)+2\beta\kappa(x_2, y_2)$ respectively. Then we can see that γ_1 coincides with γ_2 and κ_1 coincides with κ_2.

§4. From now on, we assume that $\alpha, \beta\neq 0$ and that gen. JTS $\mathfrak{J}$ of 2nd order and the induced LTS $\mathfrak{J}^*$ are non-trivial, i.e. $\{\mathfrak{J}\mathfrak{J}\mathfrak{J}\}\neq\{0\}$ and $[\mathfrak{J}^*\mathfrak{J}^*\mathfrak{J}^*]\neq\{0\}$.

For an element $X=e_1\otimes x+e_2\otimes y$ in $W\otimes\mathfrak{J}$, we define the projections P_1 and P_2 of $W\otimes\mathfrak{J}$ onto $\mathfrak{J}$ by $P_1(X)=x$ and $P_2(X)=y$ respectively. And, an involutive automorphism σ in $W\otimes\mathfrak{J}$ is defined by $\sigma(e_1\otimes x+e_2\otimes y)=e_1\otimes x-e_2\otimes y$ which induces an involutive automorphism in the LTS $(W\otimes\mathfrak{J})^*$. From the property of the product we have

Lemma 2. *Let $\mathfrak{K}$ be an ideal in $W\otimes\mathfrak{J}$ (resp. $(W\otimes\mathfrak{J})^*$), then $P_1(\mathfrak{K})$ and $P_2(\mathfrak{K})$ are ideals in $\mathfrak{J}$ (resp. $\mathfrak{J}^*$).*

From the property of the projections, we have

Lemma 3. *Let $\mathfrak{K}$ be an σ-stable ideal in $W\otimes\mathfrak{J}$ or $(W\otimes\mathfrak{J})^*$, then*

$$\mathfrak{K}=e_1\otimes P_1(\mathfrak{K})+e_2\otimes P_2(\mathfrak{K}).$$

Lemma 4. *Let $\mathfrak{K}$ be an ideal in $W\otimes\mathfrak{J}$ or $(W\otimes\mathfrak{J})^*$. If $\mathfrak{J}$ (resp. $\mathfrak{J}^*$) is simple, then $\mathfrak{K}=\{0\}$ or $P_1(\mathfrak{K})=P_2(\mathfrak{K})=\mathfrak{J}$ (resp. $\mathfrak{J}^*$).*

Using the property of σ-stable ideals, we have

Lemma 5. *Let $\mathfrak{J}$ (resp. $\mathfrak{J}^*$) be simple, then*

(i) *$W\otimes\mathfrak{J}$ (resp. $(W\otimes\mathfrak{J})^*$) is simple,*

or

(ii) *$W\otimes\mathfrak{J}$ (resp. $(W\otimes\mathfrak{J})^*$) is a direct sum of two isomorphic simple ideals in $W\otimes\mathfrak{J}$ (resp. $(W\otimes\mathfrak{J})^*$).*

Theorem 5. (i) *Let $\mathfrak{K}$ be an ideal in gen. JTS $\mathfrak{J}$ of 2nd order. Then*

$$\mathfrak{S}(W, \mathfrak{K})=id_W\otimes D(\mathfrak{K}, \mathfrak{J})\oplus l(e_1, e_2)\otimes D(\mathfrak{K}, \mathfrak{J})\oplus(W\otimes\mathfrak{K})^*$$

is an ideal in the Lie algebra $\mathfrak{S}(W, \mathfrak{J})$.

(ii) *Let $\mathfrak{K}$ be an ideal in the induced LTS $\mathfrak{J}^*$. Then*

$$\mathfrak{S}(W, \mathfrak{K})=id_W\otimes D(\mathfrak{K}, \mathfrak{J})\oplus l(e_1, e_2)\otimes D(\mathfrak{K}, \mathfrak{J})\oplus(W\otimes\mathfrak{K})^*$$

is an ideal in the Lie algebra $\mathfrak{S}(W, \mathfrak{J})$.

Corollary. *If $\mathfrak{S}(W, \mathfrak{J})$ is simple, then $\mathfrak{J}$ and $\mathfrak{J}^*$ are simple.*

§5. **Examples.** In this section we assume that the characteristic of Φ is 0 and Φ is an algebraically closed field.

(i) Let $\mathfrak{J}$ be an n-dimensional vector space with a symmetric bilinear form $\langle\ ,\ \rangle$. Then $\{xyz\}=\langle y, z\rangle x$ is a gen. Jordan triple product of 2nd order in $\mathfrak{J}$. Since the induced LT product $[xyz]$ equals to $2\langle y, z\rangle x-2\langle z, x\rangle y$, $D(\mathfrak{J}, \mathfrak{J})=\{D: \langle Dx, y\rangle+\langle x, Dy\rangle=0\}$. If the form $\langle\ ,\ \rangle$ is non-degenerate, then $\dim D(\mathfrak{J}, \mathfrak{J})=(1/2)(n^2-n)$ and $(W\otimes\mathfrak{J})^*$ is simple. Hence $\dim\mathfrak{S}(W, \mathfrak{J})=n^2+n$ and $\mathfrak{S}(W, \mathfrak{J})\cong B_l\oplus B_l (n=2l)$ or $D_l\oplus D_l (n=2l+1)$.

(ii) The quaternion algebra $\boldsymbol{Q}$ becomes a gen. JTS of 2nd order relative to a triple product $\{xyz\}=x(\bar{y}z)+z(\bar{y}x)-y(\bar{x}z)$ (cf. [1,4]). By direct calculations, we see that $\dim D(\boldsymbol{Q}, \boldsymbol{Q})=3$ and $(W\otimes\boldsymbol{Q})^*$ is simple. Hence $\dim \mathfrak{G}(W, \boldsymbol{Q})=14$ and $\mathfrak{G}(W, \boldsymbol{Q})$ is of type G_2.

(iii) The Cayley algebra $\mathfrak{C}$ becomes a gen. JTS of 2nd order relative to the same triple product as in (ii). By straightforward calculations, we see that $\dim D(\mathfrak{C}, \mathfrak{C})=7$ and $(W\otimes\mathfrak{C})^*$ is simple. Hence $\dim \mathfrak{G}(W, \mathfrak{C})=30$ and $\mathfrak{G}(W, \mathfrak{C})\cong A_3\oplus A_3\cong D_3\oplus D_3$.

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