43. Zeta Functions in Several Variables Associated with Prehomogeneous Vector Spaces. III*)

Eisenstein Series for Indefinite Quadratic Forms

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In the present note, by applying the general theory developed in [2], we prove functional equations of Eisenstein series for indefinite quadratic forms.

6. Let Y be an $n+1$ by $n+1$ rational non-degenerate symmetric matrix of signature (p, q) $(p+q=n+1)$. Denote by $d_i(A)$ the determinant of the upper left i by i block of a matrix A. Let Γ_∞ be the group of upper triangular integral matrices of size $n+1$ with diagonal entries 1. For an $n+1$ tuple $\varepsilon=(\varepsilon_1, \cdots, \varepsilon_{n+1})$ of ± 1, we write $\operatorname{sgn}\varepsilon=(i, n-i+1)$ if exactly i of ε_j's are equal to 1. For any $\varepsilon\in\{\pm 1\}^{n+1}$ with $\operatorname{sgn}\varepsilon=(p, q)$, the Eisenstein series for Y is defined by

$$E(Y, \varepsilon; s)=\sum_U \prod_{i=1}^n |d_i({}^tUYU)|^{-s_i} \quad (s=(s_1, \cdots, s_n)\in \boldsymbol{C}^n)$$

where U runs through a set of all representatives of the double cosets belonging to $SO(Y)_{\boldsymbol{Z}}\backslash SL(n+1)_{\boldsymbol{Z}}/\Gamma_\infty$ such that

$$d_i({}^tUYU)/|d_i({}^tUYU)|=\varepsilon_1\cdots\varepsilon_i \quad (1\leqq i\leqq n+1).$$

Let $z=(z_1, \cdots, z_{n+1})$ be a variable which is connected to s by $s_i=z_{i+1}-z_i+1/2$ $(1\leqq i\leqq n)$. Set

$$\Lambda(Y, \varepsilon; z)=\sum_{1\leqq j<i\leqq n+1} \eta(2z_i-2z_j+1)|\det Y|^{z_{n+1}}E(Y, \varepsilon; s)$$

where $\eta(z)=\pi^{-z/2}\Gamma(z/2)\zeta(z)$ ($\zeta(z)$: the Riemann zeta function).

Theorem 6. (1) *The series $E(Y, \varepsilon; s)$ $(\varepsilon\in\{\pm 1\}^{n+1}, \operatorname{sgn}\varepsilon=(p, q))$ are absolutely convergent for $\operatorname{Re} s_1, \cdots, \operatorname{Re} s_n>1$.*

(2) *The functions $E(Y, \varepsilon; s)$ multiplied by*

$$\prod_{1\leqq i\leqq j\leqq n}\left(s_i+s_{i+1}+\cdots+s_j-\frac{j-i}{2}-1\right)^2\zeta(2(s_i+s_{i+1}+\cdots+s_j)-j+i)$$

have analytic continuations to entire functions of s in $\boldsymbol{C}^n$.

(3) *For any permutation σ in $n+1$ letters and for any ε, $\eta\in\{\pm 1\}^{n+1}$ such that $\operatorname{sgn}\varepsilon=\operatorname{sgn}\eta=(p, q)$, there exists $A^\sigma(\varepsilon, \eta; z)$ a rational function of trigonometric functions of z satisfying*

$$\Lambda(Y, \varepsilon; \sigma z)=\sum_\eta A^\sigma(\varepsilon, \eta; z)\Lambda(Y, \eta; z)$$

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where $\sigma z=(z_{\sigma(1)}, \cdots, z_{\sigma(n+1)})$.

(4) *For the cyclic permutation* $\sigma=(k+1, 1, 2, \cdots, k)$ $(1\leqq k\leqq n)$,

$$A^{\sigma}(\varepsilon, \eta; z)=\begin{cases} \prod_{i=1}^{k} \dfrac{\cos\dfrac{\pi}{4}\left\{2(1+\varepsilon_{i+1}\eta_i)\left(z_{k+1}-z_i+\dfrac{1}{2}\right)+\varepsilon_{i+1}\left(\sum_{j=i+2}^{k+1}\varepsilon_j-\sum_{j=i}^{k+1}\eta_j\right)\right\}}{\sin\pi(z_{k+1}-z_i+1/2)} \\ \qquad \text{if } \operatorname{sgn}\varepsilon=\operatorname{sgn}\eta \text{ and } \varepsilon_i=\eta_i\ (k+2\leqq i\leqq n+1), \\ 0 \qquad \text{otherwise.} \end{cases}$$

Remarks. (1) If Y is positive definite, the series $E(Y, \varepsilon; s)$ $(\varepsilon=(1, 1, \cdots, 1))$ is the Eisenstein series of $SL(n+1)_Z$ (Selberg's zeta function) and our result is consistent with the results in A. Selberg [3] and H. Maass [1].

(2) In [3], A. Selberg suggested that one can associate with a rational indefinite quadratic form a system of Dirichlet series with functional equations similar to those of the original Eisenstein series.

7. The Eisenstein series $E(Y, \varepsilon; s)$ is a typical example of zeta functions associated with prehomogeneous vector spaces. Put $G=SO(Y)\times GL(n)\times GL(n-1)\times\cdots\times GL(1)$ and $V_k=M(k+1, k; C)$ $(1\leqq k\leqq n)$. We define a rational representation ρ_k of G on V_k by setting

$$\rho_k(g)x_k=g_{k+1}x_k g_k^{-1} \quad (g=(g_{n+1}, g_n, \cdots, g_1)\in G,\ x_k\in V_k).$$

Set $\rho=\bigoplus_{k=1}^{n}\rho_k$ and $V=\bigoplus_{k=1}^{n}V_k$.

Lemma 7. (i) *The triple* (G, ρ, V) *is a p.v. with the singular set*

$$S=\bigcup_{i=1}^{n}\{x\in V; P_i(x)=0\}$$

where $P_i(x)=\det\{{}^t(x_n x_{n-1}\cdots x_i)Y(x_n x_{n-1}\cdots x_i)\}$ $(1\leqq i\leqq n,\ x=(x_n, x_{n-1}, \cdots, x_1)\in V)$.

(ii) *For any non-empty subset* I *of* $\{1, 2, \cdots, n\}$, *put* $V_I=\oplus_{k\in I}V_k$. *Then* V_I *is a* $\boldsymbol{Q}$*-regular subspace of* (G, ρ, V) *with respect to a natural* $\boldsymbol{Q}$*-structure.*

(iii) $X_\rho(G)$ *is the group generated by* $\det g_1^2, \cdots, \det g_n^2$ *and the group* H *introduced in* [2. II] *is given by*

$$H=SO(Y)\times SL(n)\times\cdots\times SL(2)\times\{1\}.$$

Moreover the condition (I) *holds for* (G, ρ, V).

(iv) *The group* H_x *is trivial for any* $x\in V-S$.

Notice that every $\boldsymbol{Q}$-irreducible component of S is absolutely irreducible.

Denote by $(G, \rho^{(I)}, V^{(I)})$ the partially dual p.v. of (G, ρ, V) with respect to V_I and by $S^{(I)}$ its singular set. If $I=\phi$, we consider $(G, \rho^{(\phi)}, V^{(\phi)})$ as (G, ρ, V). By an easy computation, we have $\delta=(1, 1, \cdots, 1)$ for $(G, \rho^{(I)}, V^{(I)})$.

Hence, by Lemma 7, Theorem 5 of [2] and Remark (3) to Theorem 5, the zeta functions associated with $(G, \rho^{(I)}, V^{(I)})$ are absolutely convergent for $\mathrm{Re}\, s_1, \cdots, \mathrm{Re}\, s_n > 1$. Now we relate $E(Y, \varepsilon; s)$ to the zeta functions for the lattice $L = M(n+1, n; \boldsymbol{Z}) \oplus M(n, n-1; \boldsymbol{Z}) \oplus \cdots \oplus M(2, 1; \boldsymbol{Z})$. We take $SO(Y)_{\boldsymbol{R}} \times GL(n)^+_{\boldsymbol{R}} \times \cdots \times GL(1)^+_{\boldsymbol{R}}$ as $G^+_{\boldsymbol{R}}$ in [2.I] where $GL(k)^+_{\boldsymbol{R}} = \{g_k \in GL(k)_{\boldsymbol{R}}; \det g_k > 0\}$. It is easy to see that the $G^+_{\boldsymbol{R}}$-orbits in $V^{(I)}_{\boldsymbol{R}} - S^{(I)}_{\boldsymbol{R}}$ are indexed by $\{\varepsilon \in \{\pm 1\}^{n+1}; \mathrm{sgn}\, \varepsilon = (p, q)\}$.

Lemma 8. *The zeta functions $\xi^{(I)}_{\varepsilon}(L; s)$ associated with $(G, \rho^{(I)}, V^{(I)})$ and L are given by the following formula:*

$$\xi^{(I)}_{\varepsilon}(L; s) = \begin{cases} |\det Y|^{n/2} \prod_{1 \leq i \leq j \leq n} \zeta(2(s_i + \cdots + s_j) - j + i) E(Y, \varepsilon; s) & \text{if } n \notin I, \\ |\det Y|^{s_1 + \cdots + s_n - n/2} \\ \quad \times \prod_{1 \leq i \leq j \leq n} \zeta(2(s_i + \cdots + s_j) - j + i) E(Y, \varepsilon; \hat{s}) & \text{if } n \in I \end{cases}$$

where $\hat{s} = (s_n, s_{n-1}, \cdots, s_1)$.

The lemma implies the first part of Theorem 6. The functional equations satisfied by $E(Y, \varepsilon; s)$ are reduced to the functional equations combining $\xi^{(I)}_{\varepsilon}(L; s)$ with $\xi^{(\phi)}_{\varepsilon}(L; s)$ given by Theorem 2 of [2]. In particular, applying Theorem 2 to the $\boldsymbol{Q}$-regular subspace V_k, we get the functional equation of $E(Y, \varepsilon; s)$ for $\sigma = (k+1, 1, 2, \cdots, k)$. It follows from Theorem 3 of [2] that $E(Y, \varepsilon; s)$ have analytic continuations to meromorphic functions of s in $\boldsymbol{C}^n$. But the proof of Theorem 6 (2) requires more effort. The detailed proof will appear elsewhere.

References

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