# 4. Global Solution of the Initial Value Problem for a Discrete Velocity Model of the Boltzmann Equation

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**Abstract.** The Broadwell model of the Boltzmann equation for a discrete velocity gas is the space regular model with six velocities. This model is investigated on the initial value problem. It is proved that a unique solution exists globally in time for the small initial data and that the solution has the bound which depends only on the initial data. A priori estimate for the solution is based on the mass conservation law.

**§1. Introduction.** Among the discrete velocity models of the Boltzmann equation, one of the simplest is the Broadwell model (space regular model with 6 velocities) which is described by the following semilinear hyperbolic system of equations (cf. [1]):

$$\frac{\partial F_i}{\partial t}+v_i\cdot \operatorname{grad}_x F_i=\frac{1}{\varepsilon}Q_i,\qquad i=1,2,\cdots,6, \tag{1.1}$$

where $F_i(t,x)$ denotes the mass density of particles with velocity $v_i$ at the time $t$ and at the position $x=(x_1,x_2,x_3)$, $\varepsilon$ is a positive constant corresponding to the mean free path, and $Q_i/\varepsilon$ represents the rate of change of $F_i$ due to binary collisions. The velocities $v_i$ are given by

$$\begin{aligned} &v_1=(v,0,0),\quad v_2=(0,v,0),\quad v_3=(0,0,v),\\ &v_{j+3}=-v_j,\quad j=1,2,3, \end{aligned} \tag{1.2}$$

where $v$ is a positive constant, and $Q_i$ have the following form:

$$Q_j=Q_{j+3}=\sum_{k=1}^{3}(F_kF_{k+3}-F_jF_{j+3}),\qquad j=1,2,3. \tag{1.3}$$

The existence theorems of global solutions to the initial value problem for the discrete Boltzmann equation have been obtained by Nishida and Mimura [4], Crandall and Tartar [5] and Cabannes [2] only for some one-dimensional models. More precisely, Nishida and Mimura obtained the global solution for the one-dimensional Broadwell model (cf. [1]) when the initial data are small in a certain sense. Its proof is based on the mass conservation law. Crandall and Tartar obtained the global solution for the one-dimensional equations of the plane regular model with 4 velocities (cf. [3]) without smallness assumption on the initial data, by combining the result of [4] with the $H$-theorem.

Cabannes showed that the method of [5] is also valid for the one-dimensional equations of the 14-velocity model (cf. [3]).

On the other hand the global existence of the solutions for the multi-dimensional models is not known except one for the plane regular model with 4 velocities [4]. In the present paper we consider the Broadwell model (1.1) and show the global existence of the solution for the small initial data. Our proof is based on the mass conservation law and a detailed estimate of the nonlinear terms $Q_i$.

§ 2. **Theorem.** We consider the system (1.1) with the initial data

$$F_i(0, x)=F_i^0(x), \quad i=1, 2, \cdots, 6, \quad x \in \boldsymbol{R}^3. \tag{2.1}$$

We assume that the initial data $F_i^0$ belong to $\mathscr{B}^1(\boldsymbol{R}^3)$ and satisfy the following conditions for some positive constants $E_0$ and $L_j$ $(j=1, 2, 3)$:

$$0 \leq F_i^0(x) \leq E_0, \quad i=1, 2, \cdots, 6, \quad x \in \boldsymbol{R}^3, \tag{2.2}$$

$$\int_{-\infty}^{\infty} (F_i^0+F_{j+3}^0)(g_{jk}^{\pm}(s\,;x))ds \leq L_1, \quad k \neq j,\ k, j=1, 2, 3, \quad x \in \boldsymbol{R}^3, \tag{2.3$_1$}$$

$$\iint_{\boldsymbol{R}^2} (F_j^0+F_{j+3}^0)(h_l(\sigma\,;x))d\sigma \leq L_2, \quad l=0, 1, 2, 3, \quad j=1, 2, 3, \quad x \in \boldsymbol{R}^3, \tag{2.3$_2$}$$

$$\iiint_{\boldsymbol{R}^3} \left(\sum_{i=1}^{6} F_i^0\right)(\xi)d\xi \leq L_3. \tag{2.3$_3$}$$

Here we denote the straight line $\xi_j-x_j \pm (\xi_k-x_k)=0$, $\xi_l-x_l=0$ by a one parameter family $\xi=g_{jk}^{\pm}(s\,;x)$, $s=\xi_j-x_j$, and the planes $\sum_{i=1}^3 (\xi_i-x_i)=0$ and $\sum_{i \neq l} (\xi_i-x_i)=\xi_l-x_l$ by two parameter families $\xi=h_0(\sigma\,;x)$, $\sigma=(\sigma_1, \sigma_2)=(\xi_1-x_1, \xi_2-x_2)$, and $\xi=h_l(\sigma\,;x)$, $\sigma=(\sigma_1, \sigma_2)=\{\xi_i-x_i\,;i \neq l\}$, respectively.

**Theorem.** *Suppose the initial data $F_i^0 \in \mathscr{B}^1(\boldsymbol{R}^3)$ satisfy the conditions* (2.2) *and* $(2.3)_{1,2,3}$ *for positive constants $E_0$ and $L_j$ satisfying*

$$L_1+2L_2\frac{E_0}{\varepsilon v}+2L_3\left(\frac{E_0}{\varepsilon v}\right)^2 \leq \frac{\varepsilon v}{6}. \tag{2.4}$$

*Then the initial value problem* (1.1), (2.1) *has a unique global solution $F_i(t, x)$ which belongs to $C^0([0, \infty)\,;\mathscr{B}^1(\boldsymbol{R}^3)) \cap C^1([0, \infty)\,;\mathscr{B}^0(\boldsymbol{R}^3))$ and satisfies the estimate*

$$0 \leq F_i(t, x) \leq 2E_0, \quad i=1, 2, \cdots, 6, \tag{2.5}$$

*for any $t \geq 0$ and $x \in \boldsymbol{R}^3$.*

The following local existence theorem is well known (see [2], for example).

**Local existence theorem.** *If the initial data $F_i^0 \in \mathscr{B}^1(\boldsymbol{R}^3)$ satisfy the condition* (2.2), *then there exists a positive constant $T_0$, which depends only on $E_0$, such that the problem* (1.1), (2.1) *has a unique solution $F_i(t, x)$ which belongs to $C^0([0, T_0]\,;\mathscr{B}^1(\boldsymbol{R}^3)) \cap C^1([0, T_0]\,;\mathscr{B}^0(\boldsymbol{R}^3))$ and is non-negative.*

Therefore the proof of Theorem is given by the standard continuation argument of the local solution if we have the following a priori estimate for the solution.

**Lemma** (a priori estimate). *Let $T$ be some constant. If the initial data $F_i^0$ satisfy the same conditions as Theorem, then the non-negative solution $F_i \in C^0([0, T]; \mathscr{B}^1(\boldsymbol{R}^3)) \cap C^1([0, T]; \mathscr{B}^0(\boldsymbol{R}^3))$ of the problem* (1.1), (2.1) *satisfies the following a priori estimate for any $t \in [0, T]$ and $x \in \boldsymbol{R}^3$*:

$$F_i(t, x) \leq 2E_0, \qquad i=1, 2, \cdots, 6. \tag{2.6}$$

**§ 3. Proof of Lemma.** Let us estimate the quantity

$$E = \max_{1\leq i\leq 6} \sup_{\substack{0\leq t\leq T \\ x\in \boldsymbol{R}^3}} F_i(t, x) \tag{3.1}$$

by the similar method of [4]. We first integrate the mass conservation law

$$\left(\sum_{i=1}^{6} F_i\right)_t + \sum_{j=1}^{3} v(F_j - F_{j+3})_{x_j} = 0 \tag{3.2}$$

over the dependence domain $\{(\tau, \xi); 0\leq \tau \leq t - \sum_{j=1}^3 |x_j - \xi_j|/v\}$ of the point $(t, x) \in [0, T] \times \boldsymbol{R}_\xi^3$. By the Green's formula, we have

$$2\sum_{j=1}^{3}\left\{\iiint_{D\cap\{\xi_j\geq x_j\}} F_j(\hat{\tau}(\xi), \xi)d\xi + \iiint_{D\cap\{\xi_j\leq x_j\}} F_{j+3}(\hat{\tau}(\xi), \xi)d\xi\right\} = \iiint_D \left(\sum_{i=1}^{6} F_i^0\right)(\xi)d\xi, \tag{3.3}$$

where we put

$$\hat{\tau}(\xi) = t - \frac{1}{v}\sum_{j=1}^{3}|x_j - \xi_j|,$$

$$D = \left\{\xi \in \boldsymbol{R}^3; \sum_{j=1}^{3}|x_j - \xi_j| \leq vt\right\}.$$

The equality (3.3) plays an important role in proving the a priori estimate for the solution.

Now integrate the first equation of (1.1) along the characteristic line $d\xi/d\tau = v_1$ from $(0, x - v_1 t)$ to $(t, x)$, we obtain

$$F_1(t, x) = F_1^0(x_1 - vt, x_2, x_3) + \frac{1}{\varepsilon v}\int_{x_1 - vt}^{x_1} (F_2F_5 + F_3F_6 - 2F_1F_4)\left(t - \frac{|x_1 - \xi_1|}{v}, \xi_1, x_2, x_3\right)d\xi_1. \tag{3.4}$$

Since $0 \leq F_i(t, x) \leq E$ in the domain $0\leq t\leq T$, $x \in \boldsymbol{R}^3$,

$$F_1(t, x) \leq F_1^0(x_1 - vt, x_2, x_3) + \frac{E}{\varepsilon v}\int_{x_1 - vt}^{x_1} (F_2 + F_3)\left(t - \frac{|x_1 - \xi_1|}{v}, \xi_1, x_2, x_3\right)d\xi_1. \tag{3.5}$$

We estimate the integral in the second term of (3.5). For this we use the equality

$$\begin{aligned}\int_{x_1 - vt}^{x_1} F_2\left(t - \frac{|x_1 - \xi_1|}{v}, \xi_1, x_2, x_3\right)d\xi_1 &= \int_{x_1 - vt}^{x_1} F_2^0\left(\xi_1, x_2 - v\left(t - \frac{|x_1 - \xi_1|}{v}\right), x_3\right)d\xi_1 \\ &+ \frac{1}{\varepsilon v}\int_{x_1 - vt}^{x_1} d\xi_1 \int_{x_2 - v(t - |x_1 - \xi_1|/v)}^{x_2} (F_1F_4 + F_3F_6 - 2F_2F_5)\end{aligned} \tag{3.6}$$

$$\left(t-\frac{\sum_{j\neq 3}|x_j-\xi_j|}{v},\ \xi_1,\ \xi_2,\ x_3\right)d\xi_2,$$

which is obtained by the integration of the second equation of (1.1). The double integral in (3.6) must be estimated carefully. From (3.4) we have

$$\begin{aligned}
&\frac{1}{\varepsilon v}\int_{x_1-vt}^{x_1}(F_1F_4+F_3F_6-2F_2F_5)\left(t-\frac{|x_1-\xi_1|}{v},\ \xi_1,\ x_2,\ x_3\right)d\xi_1\\
&\quad=\frac{1}{2}F_1^0(x_1-vt,\ x_2,\ x_3)-\frac{1}{2}F_1(t,\ x)\\
&\qquad+\frac{3}{2\varepsilon v}\int_{x_1-vt}^{x_1}(F_3F_6-F_2F_5)\left(t-\frac{|x_1-\xi_1|}{v},\ \xi_1,\ x_2,\ x_3\right)d\xi_1\\
&\quad\le\frac{1}{2}F_1^0(x_1-vt,\ x_2,\ x_3)\\
&\qquad+\frac{3E}{2\varepsilon v}\int_{x_1-vt}^{x_1}F_3\left(t-\frac{|x_1-\xi_1|}{v},\ \xi_1,\ x_2,\ x_3\right)d\xi_1,
\end{aligned}\tag{3.7}$$

where we use $0\le F_i(t,x)\le E$. Change the variables $t\to t-|x_2-\xi_2|/v$, $x_2\to\xi_2$ in (3.7) and integrate it with respect to $\xi_2\in[x_2-vt,\ x_2]$. The resulting inequality can be written in the form

$$\begin{aligned}
&\frac{1}{\varepsilon v}\int_{x_1-vt}^{x_1}d\xi_1\int_{x_2-v(t-|x_1-\xi_1|/v)}^{x_2}(F_1F_4+F_3F_6-2F_2F_5)\\
&\qquad\qquad\left(t-\frac{\sum_{j\neq 3}|x_j-\xi_j|}{v},\ \xi_1,\ \xi_2,\ x_3\right)d\xi_2\\
&\quad\le\frac{1}{2}\int_{x_1-vt}^{x_1}F_1^0\left(\xi_1,\ x_2-v\left(t-\frac{|x_1-\xi_1|}{v}\right),\ x_3\right)d\xi_1\\
&\qquad+\frac{3E}{2\varepsilon v}\int_{x_1-vt}^{x_1}d\xi_1\int_{x_2-v(t-|x_1-\xi_1|/v)}^{x_2}F_3\\
&\qquad\qquad\left(t-\frac{\sum_{j\neq 3}|x_j-\xi_j|}{v},\ \xi_1,\ \xi_2,\ x_3\right)d\xi_2,
\end{aligned}\tag{3.8}$$

if we change the variable $\xi_2\to\xi_1=x_1-v(t-|x_2-\xi_2|/v)$ in the single integral and apply the Fubini's theorem to the double integrals.

We proceed to estimate the double integral in the right member of (3.8). We use the following equality which is obtained by the integration of the third equation of (1.1):

$$\begin{aligned}
&\int_{x_1-vt}^{x_1}d\xi_1\int_{x_2-v(t-|x_1-\xi_1|/v)}^{x_2}F_3\left(t-\frac{\sum_{j\neq 3}|x_j-\xi_j|}{v},\ \xi_1,\ \xi_2,\ x_3\right)d\xi_2\\
&\quad=\int_{x_1-vt}^{x_1}d\xi_1\int_{x_2-v(t-|x_1-\xi_1|/v)}^{x_2}F_3^0\\
&\qquad\qquad\left(\xi_1,\ \xi_2,\ x_3-v\left(t-\frac{\sum_{j\neq 3}|x_j-\xi_j|}{v}\right)\right)d\xi_2\\
&\qquad+\frac{1}{\varepsilon v}\int_{x_1-vt}^{x_1}d\xi_1\int_{x_2-v(t-|x_1-\xi_1|/v)}^{x_2}d\xi_2
\end{aligned}\tag{3.9}$$

$$\int_{x_3-v(t-\sum_{j\neq 3}|x_j-\xi_j|/v)}^{x_3}(F_1F_4+F_2F_5-2F_3F_6)(\hat{\tau}(\xi),\ \xi)d\xi_3.$$

By the equality (3.3) the triple integral in (3.9) is bounded by

$$\frac{E}{\varepsilon v}\iiint_{\bigcap_{j=1}^{3}\{\xi_j\leq x_j\}\cap D}(F_4+F_5)(\hat{\tau}(\xi),\ \xi)d\xi\leq\frac{E}{2\varepsilon v}\iiint_D\left(\sum_{i=1}^{6}F_i^0\right)(\xi)d\xi.$$

Therefore from (3.6), (3.8) and (3.9) we arrive at

$$\begin{aligned}(3.10)\quad &\int_{x_1-vt}^{x_1}F_2\left(t-\frac{|x_1-\xi_1|}{v},\ \xi_1,\ x_2,\ x_3\right)d\xi_1\\ &\leq\int_{x_1-vt}^{x_1}\left(\frac{1}{2}F_1^0+F_2^0\right)\left(\xi_1,\ x_2-v\left(t-\frac{|x_1-\xi_1|}{v}\right),\ x_3\right)d\xi_1\\ &+\frac{3}{2}\frac{E}{\varepsilon v}\int_{x_1-vt}^{x_1}d\xi_1\int_{x_2-v(t-|x_1-\xi_1|/v)}^{x_2}F_3^0\\ &\qquad\left(\xi_1,\ \xi_2,\ x_3-v\left(t-\frac{\sum_{j\neq 3}|x_j-\xi_j|}{v}\right)\right)d\xi_2\\ &+\frac{3}{4}\left(\frac{E}{\varepsilon v}\right)^2\iiint_D\left(\sum_{i=1}^{6}F_i^0\right)(\xi)d\xi.\end{aligned}$$

By the assumptions $(2.3)_{1,2,3}$ on the initial data, the right member of (3.10) has a bound

$$(3.11)\qquad \frac{3}{2}L_1+\frac{3}{2}L_2\frac{E}{\varepsilon v}+\frac{3}{4}L_3\left(\frac{E}{\varepsilon v}\right)^2.$$

In the same way, we can show that the integral

$$\int_{x_1-vt}^{x_1}F_3\left(t-\frac{|x_1-\xi_1|}{v},\ \xi_1,\ x_2,\ x_3\right)d\xi_1$$

is also bounded by (3.11). Thus the bound of the integral in (3.5) is obtained. Noting that $F_1^0(x_1-vt,\ x_2,\ x_3)\leq E_0$ in (3.5), we obtain the following estimate for any $t\in[0,\ T]$ and $x\in\boldsymbol{R}^3$:

$$F_1(t,\ x)\leq E_0+3L_1\frac{E}{\varepsilon v}+3L_2\left(\frac{E}{\varepsilon v}\right)^2+\frac{3}{2}L_3\left(\frac{E}{\varepsilon v}\right)^3.$$

Similarly we can show the same estimate for the other densities $F_i(t,\ x)$, $i=2,\ 3,\ \cdots,\ 6$. This and the definition of $E$ give the inequality

$$(3.12)\qquad E\leq E_0+3L_1\frac{E}{\varepsilon v}+3L_2\left(\frac{E}{\varepsilon v}\right)^2+\frac{3}{2}L_3\left(\frac{E}{\varepsilon v}\right)^3.$$

From this inequality we can deduce $E\leq 2E_0$, if the constants $E_0$ and $L_j$ satisfy (2.4). This completes the proof of Lemma.

## References

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