39. *Moduli of Anti-Self-Dual Connections on Kähler Manifolds*

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0. The aim of this note is to give a brief proof of the following theorem which concerns the moduli space of anti-self-dual connections on a Kähler manifold.

Theorem. *Let M be a compact Kähler 2-manifold of positive scalar curvature. Let P be a G-principal bundle over M, and E a complex vector bundle associated to P, where G is a compact semi-simple Lie group. If a G-connection on E is anti-self-dual and irreducible, then the space of its infinitesimal deformations is of dimension*

$$-Pont_1(\mathfrak{g}_P^C)-\frac{1}{2}\dim G(\chi+\tau),$$

where $Pont_1(\mathfrak{g}_P^C)$ is the first Pontrjagin class of the adjoint bundle $\mathfrak{g}_P^C$, χ and τ are the Eular number and the signature of M respectively.

In the case when M is a self-dual Riemannian 4-manifold, the following remarkable theorem is obtained by Atiyah, Hitchin and Singer.

Theorem (Atiyah, Hitchin and Singer [1]). *Let M be a compact oriented self-dual Riemannian 4-manifold of positive scalar curvature. Let P be a G-principal bundle, where G is a compact semi-simple Lie group. Then, the space of moduli of irreducible self-dual connections on P is either empty or a manifold of dimension*

$$Pont_1(\mathfrak{g}_P^C)-\frac{1}{2}\dim G(\chi-\tau).$$

Here an oriented Riemannian 4-manifold is called self-dual if its Weyl's conformal curvature tensor is self-dual.

The proof of this theorem was proceeded with three parts as follows; (1) to compute the dimension of the space of infinitesimal deformations by using the Atiyah-Singer index theorem together with a vanishing theorem, (2) to use the method of Kuranishi in order to obtain a local homeomorphism from the space to a local moduli space and (3) to show that these local spaces give local charts on the global moduli space.

While the dimension of the space of infinitesimal deformations in our theorem is calculated in a similar manner to the method used in their proof, a vanishing theorem is directly verified without using the Dirac operator.

By the aid of the second and third parts in their proof, we have the following

Corollary. *With the same assumption as in our theorem, the space of anti-self-dual irreducible G-connections is a manifold of dimension*

$$-Pont_1(\mathfrak{g}_P^C)-\frac{1}{2}\dim G(\chi+\tau),$$

if it is not empty.

Remark 1. It is shown that a Kähler 2-manifold is self-dual if and only if it is a complex space form. Thus, $P_1(C)\times P_1(C)$ admits a Kähler metric of positive scalar curvature, whereas it is not self-dual.

Remark 2. If an orientation of the base space is reversed, then, with respect to the reversed orientation, an anti-self-dual connection is self-dual and $Pont_1(\cdot)$ and τ have the inverse sign. Therefore, the dimension of the moduli space of self-dual irreducible G-connections on a Kähler 2-manifold of positive scalar curvature is in agreement with the value given in the theorem of Atiyah, Hitchin and Singer.

Remark 3. Let E be a G-complex vector bundle over a compact Kähler 2-manifold of the second Betti number 1. If E admits an anti-self-dual G-connection, then the Chern classes satisfy $c_1(E)=0$ and $c_2(E)=n\geqq 0$. The dimension of the moduli space is $2(4n-3)$ in the case of $M=P_2(C)$ and $G=SU(2)$. It is known, on the other hand, that the moduli of algebraic rank 2 vector bundle over $P_2(C)$ with $c_1=0$ and $c_2=n$ is a manifold of complex dimension $4n-3$ ([3]).

A paper which deals with a full description of the results stated in this note will be published elsewhere ([5]).

1. Infinitesimal deformations. Let M be a compact orientable Riemannian 4-manifold and P a G-principal bundle over M, where G is a compact semi-simple Lie group. A faithful representation of G to $GL(V)$ with respect to a complex vector space V induces a complex vector bundle $E=P\times_G V$ with structure group G. A G-connection ∇ on E is called anti-self-dual if R^∇ satisfies $*R^\nabla=-R^\nabla$, where R^∇ is the curvature form of ∇ defined as $R^\nabla(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$, and $*$ is the Hodge star operator defined by a fixed orientation. If $\mathfrak{g}_P$ denotes the adjoint bundle $P\times_{(G,Ad)}\mathfrak{g}$, then R^∇ is a $\mathfrak{g}_P$-valued 2-form.

Now assume that M is a Kähler 2-manifold. By the orientation defined by the complex structure of M, we have that a 2-form ω is anti-self-dual, that is, $*\omega=-\omega$, if and only if it is type (1,1) and is orthogonal to Ω ([1]). Here Ω is the fundamental form induced from the Kähler metric g. Thus, the condition that a G-connection ∇ is anti-self-dual is

(1.1) $$R^{\nabla} \in \Gamma(M; \Lambda^{1,1} \otimes \mathfrak{g}_P), \qquad \langle R^{\nabla}, \Omega \rangle = 0.$$

Here and in what follows, we use the following notation; $A^k(F) = \Gamma(M; \Lambda^k \otimes F)$, $A^k_C(F) = \Gamma(M; \Lambda^k_C \otimes F)$ and $A^{p,q}(F) = \Gamma(M; \Lambda^{p,q} \otimes F)$ where Λ^k and Λ^k_C are the bundles of k-forms and its complexification and $\Lambda^{p,q}$ denotes the bundle of (p, q)-forms.

The connection ∇ is naturally defined on the subbundle $\mathfrak{g}_P \subset End(E)$ and the exterior covariant derivative d^{∇}; $A^k_C(\mathfrak{g}_P) \to A^{k+1}_C(\mathfrak{g}_P)$ splits into $\partial^{\nabla} + \bar{\partial}^{\nabla}$, where ∂^{∇}; $A^{p,q}(\mathfrak{g}_P) \to A^{p+1,q}(\mathfrak{g}_P)$ and $\bar{\partial}^{\nabla}$; $A^{p,q}(\mathfrak{g}_P) \to A^{p,q+1}(\mathfrak{g}_P)$. By the definition of the curvature form R^{∇}, we have $d^{\nabla} \circ d^{\nabla}(\Psi) = [\Psi \wedge R^{\nabla}]$ for $\Psi \in A^k(\mathfrak{g}_P)$. Thus, if R^{∇} is of type (1, 1), then

$$d^{\nabla} \circ d^{\nabla}(A^{p,q}(\mathfrak{g}_P)) \subset A^{p+1,q+1}(\mathfrak{g}_P).$$

Hence, we have

(1.2) $$\partial^{\nabla} \circ \partial^{\nabla} = 0, \quad \bar{\partial}^{\nabla} \circ \bar{\partial}^{\nabla} = 0, \quad (\partial^{\nabla} \circ \bar{\partial}^{\nabla} + \bar{\partial}^{\nabla} \circ \partial^{\nabla})\Psi = [\Psi \wedge R^{\nabla}].$$

Now suppose that ∇^t ($|t| < \varepsilon$) is a one-parameter family of anti-self-dual G-connections with $\nabla^0 = \nabla$. Then we have $\nabla^t = \nabla + tA + o(t)$ with respect to the infinitesimal deformation $A = (d/dt)\nabla^t|_{t=0} \in A^1(\mathfrak{g}_P)$. Since R^{∇^t} satisfies (1.1), we have

(1.3) $$\partial^{\nabla} A^+ = 0, \qquad \langle \bar{\partial}^{\nabla} A^+ + \partial^{\nabla} \overline{A^+}, \Omega \rangle = 0,$$

where $A = A^+ + \overline{A^+}$, A^+ is a $\mathfrak{g}^C_P$-valued 1-form of type (1, 0).

Put $\Lambda^2_+ = (\Lambda^{2,0} + \Lambda^{0,2})_{\mathbf{R}} + \mathbf{R}\Omega$ where $(\Lambda^{2,0} + \Lambda^{0,2})_{\mathbf{R}} = \{\omega + \bar{\omega}; \omega \in \Lambda^{2,0}\}$ and $A^2_+(\mathfrak{g}_P) = \Gamma(M; \Lambda^2_+ \otimes \mathfrak{g}_P)$. Now define a first order differential operator δ^{∇}; $A^1(\mathfrak{g}_P) \to A^2_+(\mathfrak{g}_P)$ by

(1.4) $$\delta^{\nabla}(A) = (\partial^{\nabla} A^+ + \bar{\partial}^{\nabla} \overline{A^+}) + \Omega \langle \bar{\partial}^{\nabla} A^+ + \partial^{\nabla} \overline{A^+}, \Omega \rangle.$$

Hence, $A \in A^1(\mathfrak{g}_P)$ is in $Ker(\delta^{\nabla})$ if and only if A is an infinitesimal deformation of ∇.

2. A complex associated to an anti-self-dual connection. Let $G_P = P \times_G G$ be the associated bundle induced from the conjugation of G. A smooth section of G_P is called a *gauge transformation* of E and $\mathcal{G}_P = \Gamma(M; G_P)$ is called the *gauge transformation group* of E. A gauge transformation acts on G-connections by $f(\nabla) = f^{-1} \circ \nabla \circ f$ for $f \in \mathcal{G}_P$ and a G-connection ∇. By a physical meaning in Yang-Mills theory, any G-connections are identified if they are transformed each other by a gauge transformation. Let f_t be a one-parameter family of gauge transformations with $f_0 = id$. The infinitesimal gauge transformation $\dot{f} = (d/dt)f_t|_{t=0}$ belongs to $A^0(\mathfrak{g}_P)$. Then $\nabla \dot{f}$ gives the infinitesimal deformation of $f_t(\nabla)$ at $t = 0$. Since $R^{f(\nabla)} = Ad(f)R^{\nabla}$, if ∇ is anti-self-dual, then so is $f_t(\nabla)$, hence $\nabla \dot{f} \in Ker(\delta^{\nabla})$. Therefore, the space of infinitesimal deformations of ∇ is nothing but the first cohomology space $H^1 = Ker\, \delta^{\nabla} / Im\, \nabla$ of the following elliptic complex

(2.1) $$0 \longrightarrow A^0(\mathfrak{g}_P) \xrightarrow{d^0 = \nabla} A^1(\mathfrak{g}_P) \xrightarrow{d^1 = \delta^{\nabla}} A^2_+(\mathfrak{g}_P) \longrightarrow 0.$$

It is not hard to verify that this is elliptic.

3. Vanishing theorem. Since (2.1) is elliptic, $H^i = Ker\, d^i/Im\, d^{i-1}$ is isomorphic to $\{\Psi \in A^i(\mathfrak{g}_P);\ D^i\Psi = 0\}$, where $D^i = d^{i*} \circ d^i + d^{i-1} \circ d^{i-1*}$ is the Laplacian, $i=0, 1, 2$ (d^{i*} denotes the formal adjoint of d^i). By using the Atiyah-Singer index theorem, the index $h^0 - h^1 + h^2$ ($h^i = \dim H^i$) can be represented universally by the characteristic classes of M and $\mathfrak{g}_P$.

In this section, we will show the following proposition which gives vanishing estimates for h^0 and h^2.

Proposition 3.1. *Let M be a compact Kähler 2-manifold of positive scalar curvature. If the anti-self-dual G-connection ∇ is irreducible, then $h^0 = h^2 = 0$. Here, that ∇ is irreducible implies that there is no proper closed subgroup of G which contains the holonomy group of ∇.*

Proof. If $f \in A^0(\mathfrak{g}_P)$ satisfies $D^0 f = 0$, then $\nabla f = 0$. Hence f commutes with the holonomy group. By the irreducibility of ∇, f belongs to the center of $\mathfrak{g}_P$. That f identically vanishes follows from the semi-simplicity of G.

We require the following lemmas in order to prove $h^2 = 0$.

Lemma 3.2. *For $\Omega \otimes f$, $f \in A^0(\mathfrak{g}_P)$, we have*

$$d^1 \circ d^{1*}(\Omega \otimes f) = 4\Omega \otimes \left(\sum_{i=1}^{2} \nabla_{\bar{e}_i} \nabla_{e_i} f\right), \tag{3.1}$$

at any point x in M, where $\{e_1, e_2\}$ is a local unitary frame of M with $De_i = 0$ at x, $i=1, 2$, where D is the Levi-Civita connection induced from the Kähler metric g.

Lemma 3.3. *With respect to the Laplacian $\Box^\nabla = \partial^\nabla \circ \partial^{\nabla *} + \partial^{\nabla *} \circ \partial^\nabla$ associated to the complex ∂^∇; $A^{p,0}(\mathfrak{g}_P) \to A^{p+1,0}(\mathfrak{g}_P)$ given by (1.2), we have*

$$\Box^\nabla \Psi(X, Y) = -\sum_i (\nabla_{\bar{e}_i} \nabla_{e_i} \Psi)(X, Y) + \Psi(\mathrm{Ric}\, X, Y) + \Psi(X, \mathrm{Ric}\, Y), \tag{3.2}$$

for $\Psi \in A^{2,0}(\mathfrak{g}_P)$ where Ric *denotes the Ricci tensor of g.*

These lemmas are verified by a slight calculation with the aid of the facts (1.1) and $d\Omega = 0$ and also of the formulas (3, 2) and (3, 10) stated in [4].

Now assume that $\Psi = \Psi^{2,0} + \overline{\Psi^{2,0}} + \Omega \otimes f \in A^2_+(\mathfrak{g}_P)$ satisfies $D^2\Psi = d^1 \circ d^{1*}\Psi = 0$. Since any elements in $A^k_C(\mathfrak{g}_P)$ of different types are orthogonal with respect to the Hermitian inner product on $A^k_C(\mathfrak{g}_P)$ induced naturally by g, we have

$$\begin{aligned} 0 &= \langle D^2\Psi, \Psi^{2,0} + \overline{\Psi^{2,0}} \rangle = \langle d^1 \circ d^{1*}(\Psi^{2,0} + \overline{\Psi^{2,0}}), \Psi^{2,0} + \overline{\Psi^{2,0}} \rangle \\ &= \langle \partial^\nabla \circ \partial^{\nabla *}\Psi^{2,0}, \Psi^{2,0} \rangle + \langle \bar{\partial}^\nabla \circ \bar{\partial}^{\nabla *}\overline{\Psi^{2,0}}, \overline{\Psi^{2,0}} \rangle, \end{aligned}$$

where we use the fact that $d^1 \circ d^{1*}(\Omega \otimes f)$ is of type (1, 1) from Lemma 3.2. Because $\partial^\nabla \Psi^{2,0} \in A^{3,0}(\mathfrak{g}_P) = \{0\}$, we have $\langle \Box^\nabla \Psi^{2,0}, \Psi^{2,0} \rangle + \langle \overline{\Box^\nabla \Psi^{2,0}}, \overline{\Psi^{2,0}} \rangle = 0$. Integrate this over M, then by using Lemma 3.3

$$2\|\partial^\nabla \Psi^{2,0}\|^2_M + 2\int_M \rho \langle \Psi^{2,0}, \Psi^{2,0} \rangle\, dv = 0,$$

here ρ is the scalar curvature. By the assumption that ρ is positive, $\Psi^{2,0}=0$, that is, $\Psi=\Omega\otimes f$. By Lemma 3.2, $0=\langle d^1\circ d^{1*}(\Omega\otimes f), \Omega\otimes f\rangle =4\langle\Omega\otimes(\sum \nabla_{\bar{e}_i}\nabla_{e_i}f), \Omega\otimes f\rangle=8\langle\sum \nabla_{\bar{e}_i}\nabla_{e_i}f, f\rangle$. Integrate this over M, then $0=\langle d^1\circ d^{1*}(\Omega\otimes f), \Omega\otimes f\rangle_M=-8\|\partial^\nabla f\|_M^2$, hence $\partial^\nabla f=0$. Since $\nabla f =\partial^\nabla f+\bar{\partial}^\nabla f=\partial^\nabla f+\overline{\partial^\nabla f}=0$, we have $f=0$ from the irreducibility of ∇, that is $h^2=0$.

4. Index of the complex. The principal symbols of complex (2.1) is

$$(4.1)\quad 0\longrightarrow\pi^*(\Lambda^0\otimes\mathfrak{g}_P)\xrightarrow{\sigma(d)\otimes id_{\mathfrak{g}_P}}\pi^*(\Lambda^1\otimes\mathfrak{g}_P)\xrightarrow{\sigma(\delta)\otimes id_{\mathfrak{g}_P}}\pi^*(\Lambda^2_+\otimes\mathfrak{g}_P)\longrightarrow 0,$$

where π denotes the projection from $T^*M\backslash\{0\}$ to M, $\sigma(d)$ and $\sigma(\delta)$ are the respective principal symbols of the following elliptic complex associated to the base space M,

$$(4.2)\quad 0\longrightarrow\Gamma(M;\Lambda^0)\xrightarrow{d}\Gamma(M;\Lambda^1)\xrightarrow{\delta}\Gamma(M;\Lambda^2_+)\longrightarrow 0.$$

Here d is the exterior derivative and δ is defined by

$$\delta(\tau^++\overline{\tau^+})=\partial\tau^++\overline{\partial\tau^+}+\langle\bar{\partial}\tau^++\partial\overline{\tau^+}, \Omega\rangle\Omega,\ \tau^+\in\Gamma(M;\Lambda^{1,0}).$$

By using Proposition 2.17 in [2], the index of the complex is given by

$$h^0-h^1+h^2=ch(\mathfrak{g}_P^C)\cdot\{ch(\Lambda_C^0)-ch(\Lambda_C^1)+ch(\Lambda_+^2)_C\}\cdot\mathcal{T}(TM\otimes C)/_{e(TM)}[M].$$

Here $e(TM)$ is the Euler class of M, ch is the Chern character and $\mathcal{T}$ is the Todd class. By a simple computation, the index is $(c_1^2-2c_2)(\mathfrak{g}_P^C)[M] +1/6\dim G(c_1^2+c_2)(M)=Pont_1(\mathfrak{g}_P^C)+1/2\dim G(\chi+\tau)$. Hence, we have $h^1=-Pont_1(\mathfrak{g}_P^C)-1/2\dim G(\chi+\tau)$. Thus, the theorem is obtained.

References

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