36. *Branching of Singularities for Degenerate Hyperbolic Operators and Stokes Phenomena.* II

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Recently, one of the authors revealed a closed connection between branching of singularities and Stokes phenomena for a certain class of degenerate hyperbolic operators ([2]). We generalize his result to the following type of degenerate hyperbolic linear partial differential operators P in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$:

$$P=\sum_{i=0}^{m} P_{m-i}(t, x, D_t, D_x),$$

$$P_m(t, x, \tau, \xi)=\prod_{i=1}^{m} (\tau - t^{\ell}\lambda_i(t, x, \xi)),$$

$$P_{m-i}(t, x, \tau, \xi)=\sum_{j=0}^{m-i} t^{\sigma(i,j)} P_{ij}(t, x, \xi)\tau^{m-i-j},$$

where $D_t=\dfrac{\partial}{\sqrt{-1}\partial t}$, $D_x=(D_{x_1}, \cdots, D_{x_n})=\left(\dfrac{\partial}{\sqrt{-1}\partial x_1}, \cdots, \dfrac{\partial}{\sqrt{-1}\partial x_n}\right)$, $\ell \in \boldsymbol{N}$, $\sigma(i, j)=\max(j\ell-i, 0)$, $\lambda_i(t, x, \xi) \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n \backslash 0, \boldsymbol{R}\backslash 0)$ are homogeneous of degree 1 with respect to ξ, and $P_{ij}(t, x, \xi) \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ are homogeneous polynomials of degree j with respect to ξ. Moreover, $\lambda_i(t, x, \xi)$ satisfy $|\lambda_i(t, x, \xi)-\lambda_j(t, x, \xi)| \geq C|\xi|$ $(t \in \boldsymbol{R}, x \in \boldsymbol{R}^n, \xi \in \boldsymbol{R}^n \backslash 0)$ for some $C>0$ if $i \neq j$.

As for P, Uryu [8] established the $\mathcal{E}$ wellposedness of the Cauchy problem and Nakamura-Uryu [4] and Shinkai [6] illustrated the construction of a backward and a forward parametrices of the Cauchy problem with initial data at $t=0$ in terms of Fourier integral operators.

In this note we show that the equation $Pu=0$ possesses a solution whose singularities branch at $t=0$. The outline of the proof is as follows. According to the construction of parametrix given by Nakamura-Uryu [4], the main parts of the amplitudes which consist the parametrix are determined by a fundamental system of solutions of the ordinary differential operator

$$L=\sum_{i=0}^{m} \sum_{j=0}^{m-i} t^{\sigma(i,j)} P_{ij}(0, x, \xi) D_t^{m-i-j}.$$

Its asymptotic expansions for large $|\xi|$ considered in $t>0$ and $t>0$ are different (namely, Stokes phenomena occurs at $t=0$). The one is different from the other by multiplying Stokes multipliers. Observing this we can show our assertion by composing the forward and backward parametrices with an appropriate initial data at $t=0$. The key point is how to choose this initial data. In the following we state our theorem and list up some lemmas which are necessary for the proof. The details of the proof and more complete results will be published elsewhere.

Let T be a sufficiently small positive constant. By $\phi_i(t, x, \xi)$ we denote the phase function such that

$$\begin{cases} \dfrac{\partial}{\partial t}\phi_i(t, x, \xi)=t^{\ell}\lambda_i(t, x, \nabla_x\phi_i(t, x, \xi)) \quad (|t|\leq T), \\ \phi_i(0, x, \xi)=x\cdot\xi. \end{cases}$$

We set $I_\sigma=\{t: 0\leq\sigma t\leq T\}$ for $\sigma=\pm 1$. By $S^{q,r}(I_\sigma)$ we denote the set of all functions $U(t, x, \xi)\in C^\infty(I_\sigma\times \boldsymbol{R}_x^n\times \boldsymbol{R}_\xi^n\backslash 0)$ such that for any nonnegative integer a and for any nonnegative integral multi-indices α, β there is a constant $C_{a,\alpha,\beta}$ satisfying

$$|D_t^a D_x^\alpha D_\xi^\beta U(t, x, \xi)|\leq C_{a,\alpha,\beta}(1+|\xi|)^{q-|\beta|}(|\xi|^{-1}+|t|^{\ell+1})^{(r-a)/(\ell+1)}$$

for all $(t, x, \xi)\in I_\sigma\times \boldsymbol{R}_x^n\times\{\xi\in \boldsymbol{R}^n: |\xi|\geq 1\}$. We also set

$$S^{q,\infty}(I_\sigma)=\bigcap_{r=-\infty}^{\infty} S^{q,r}(I_\sigma), \qquad S^{-\infty}(I_\sigma)=\bigcap_{q=-\infty}^{\infty} S^{q,r}(I_\sigma),$$

$$S^{q+0,r+0}(I_\sigma)=\bigcap_{\varepsilon>0} S^{q+\varepsilon,r+\varepsilon}(I_\sigma), \qquad S^{q+0,\infty}(I_\sigma)=\bigcap_{r=-\infty}^{\infty}\bigcap_{\varepsilon>0} S^{q+\varepsilon,r}(I_\sigma).$$

For simplicity, we use the variables $w=t|\xi|^{1/(\ell+1)}$, $z=(t^{\ell+1}/(\ell+1))|\xi|$, $\theta=|\xi|^{-1}\xi$.

Lemma 1. *The equation $Lv=0$ has formal solutions $v_i(z, x, \theta)$ $(1\leq i\leq m)$ of the forms*

$$v_i(z, x, \theta)=\exp\{\sqrt{-1}\lambda_i(0, x, \theta)z\}z^{\mu_i(x,\theta)}\sum_{\nu=0}^{\infty} c_i(\nu, x, \theta)z^{-\nu}$$

where

$$\mu_i(x, \theta)=-\sum_{j=1}^{m}\left\{\frac{(m-j+1)(m-j)\ell}{2(\ell+1)}P_{0,j-1}(0, x, \theta)+\frac{\sqrt{-1}}{\ell+1}P_{1,j-1}(0, x, \theta)\right\} \times\frac{\lambda_i^{m-j}(0, x, \theta)}{\prod_{k=1,k\neq i}^{m}(\lambda_i(0, x, \theta)-\lambda_k(0, x, \theta))}$$

and $c_i(\nu, x, \theta)$ $(\nu\geq 0)$ are determined as in [2].

Lemma 2. *There is a fundamental system of solutions $V_{\sigma i}(w, x, \theta)$ $(1\leq i\leq m, \sigma=\pm 1)$ of the equation $LV=0$ such that $V_{\sigma i}(w, x, \theta)\sim v_i(z, x, \theta)$ as $\sigma w\to+\infty$ in $\boldsymbol{R}$. Here "$\sim$" stands for the asymptotic equality for all the derivatives of $V_{\sigma i}(w, x, \theta)$ and $v_i(z, x, \theta)$ provided that we mean by the derivatives of $v_i(z, x, \theta)$ the differentiation of $v_i(z, x, \theta)$ in the termwise sense.*

Lemma 3. *There are Stokes multipliers $S_{\sigma_1 i,\sigma_2 j}(x, \theta)$ $(1\leq i, j\leq m$, $\sigma_1, \sigma_2=\pm 1)$ such that $S_{\sigma_1 i,\sigma_2 j}(x, \theta)\in C^\infty(\boldsymbol{R}_x^n\times S_\theta^{n-1})$ and*

$$V_{\sigma_2 j}(w, x, \theta)=\sum_{i=1}^{m} V_{\sigma_1 i}(w, x, \theta) S_{\sigma_1 i, \sigma_2 j}(x, \theta).$$

We set

$$W_{\sigma i}^{k}(w, x, \theta)=\exp\{-\sqrt{-1}\lambda_i(0, x, \theta) z\}\left(\frac{\partial}{\partial t}\right)^{k} V_{\sigma i}(w, x, \theta)$$

$$D_\sigma(x, \theta)=\det\left(\left(\frac{\partial}{\partial w}\right)^{k} V_{\sigma i}(0, x, \theta): \begin{matrix} k \downarrow 0, \cdots, m-1 \\ i \rightarrow 1, \cdots, m \end{matrix}\right)$$

and by $\tilde{W}_{\sigma i}^{k}(w, x, \theta)$ we denote the (k, i)-cofactor of the matrix

$$\left(W_{\sigma i}^{k}(w, x, \theta): \begin{matrix} k \downarrow 0, \cdots, m-1 \\ i \rightarrow 1, \cdots, m \end{matrix}\right).$$

Lemma 4.

$$W_{\sigma i}^{k}(w, x, \theta) \sim \sqrt{-1}^{k} t^{k\ell} |\xi|^{k} z^{\mu_i(x, \theta)}\{\lambda_i^k(0, x, \theta)+0(z^{-1})\}$$

$$\tilde{W}_{\sigma i}^{k}(w, x, \theta) \sim \sqrt{-1}^{m(m-1)/2-k}(\ell+1)^{m(m-1)\ell/2(\ell+1)} t^{-k\ell}$$
$$\times|\xi|^{m(m-1)/2(\ell+1)-k} z^{-\mu_i(x, \theta)}\{\tilde{\lambda}_i^k(0, x, \theta)+0(z^{-1})\}$$

as $\sigma w \rightarrow +\infty$ *in* $\boldsymbol{R}$. *Here* $\tilde{\lambda}_i^k(0, x, \theta)$ *is the* (k, i)-*cofactor of the matrix* $\left(\lambda_i^k(0, x, \theta): \begin{matrix} k \downarrow 0, \cdots, m-1 \\ i \rightarrow 1, \cdots, m \end{matrix}\right)$ *and* $0(z^{-1})$ *denotes a* C^∞ *function* $f_\sigma(z, x, \theta)$ *defined in* $\sigma w>0$, $x \in \boldsymbol{R}^n$, $|\theta|=1$ *such that*

$$|D_z^a D_x^\alpha D_\theta^\beta f(z, x, \theta)| \leq C_{a, \alpha, \beta}|z|^{-1-a} \quad (\sigma w \geq 1, x \in \boldsymbol{R}^n, |\theta|=1)$$

for any nonnegative integer a *and any nonnegative multi-indices* α, β *and for some* $C_{a, \alpha, \beta}>0$.

We define the transport operators T_i, $T_{h,i}$ by

$$T_i=\exp\{-\sqrt{-1}\phi_i\} P(\exp\{\sqrt{-1}\phi_i\}\cdot),$$

$$T_{h,i}=\exp\{-\sqrt{-1}\phi_i\}\left(\frac{\partial}{\partial t}\right)^{h}(\exp\{\sqrt{-1}\phi_i\}\cdot).$$

We set $\nu_i=\sup_{(x, \theta) \in \boldsymbol{R}_x^n \times S_\theta^{n-1}} \operatorname{Re} \mu_i(x, \theta)$.

Lemma 5 (Nakamura-Uryu [4]). *There exist symbols*

$$U_{\sigma i, k}(t, x, \xi) \in S^{\nu_i-(k/\ell+1)+0, \nu_i(\ell+1)+0}(I_\sigma), \quad U_{\sigma i, k}^{*}(t, x, \xi) \in S^{\nu_i-(k/\ell+1)+0, \infty}(I_\sigma)$$

such that

$$\begin{cases} T_i(U_{\sigma i, k}+U_{\sigma i, k}^{*})=0 \\ \sum_{i=1}^{m} T_{h, i}(U_{\sigma i, k}+U_{\sigma i, k}^{*})|_{t=0}=\delta_{hk} \quad (0 \leq h \leq m-1) \end{cases}$$

$\bmod S^{-\infty}(I_\sigma)$. *Furthermore,* $U_{\sigma i, k}(t, x, \xi)$ *admits an asymptotic expansion*

$$U_{\sigma i, k} \sim \sum_{\nu=0}^{\infty} U_{\sigma i, k}^{\nu}$$

(namely, $U_{\sigma i, k}-\sum_N U_{\sigma i, k}^{\nu} \in S^{\nu_i-(k/\ell+1)+0, \nu_i(\ell+1)+N+0}(I_\sigma)$ for $N=1, 2, \cdots$). *Here* $U_{\sigma i, k}^{0,0}(t, x, \xi)$ *is given by*

$$U_{\sigma i, k}^{0,0}(t, x, \xi)=W_{\sigma i}(t|\xi|^{1/(\ell+1)}, x, \theta) W_{\sigma i}^{k}(0, x, \theta)$$
$$\times|\xi|^{-m(m-1)/2(\ell+1)} D^{-1}(x, \theta) \in S^{\nu_i-k/(\ell+1)+0, \nu_i(\ell+1)+0}(I_\sigma).$$

Fix $(x^0, \theta^0) \in \boldsymbol{R}_x^n \times S_\theta^{n-1}$ and define the Fourier integral operator $F_{\sigma i, -j}(t): \mathcal{S}' \rightarrow \mathcal{S}'$ by

$$F_{\sigma i,-j}(t)f(x)=\sum_{k=0}^{m-1}\int \exp\{\sqrt{-1}\phi_i(t,x,\xi)\}\chi(\xi)(U_{\sigma i,k}+U^*_{\sigma i,k})(t,x,\xi)$$
$$\times W^k_{-j}(0,x^0,\theta)\hat{f}(\xi)d\xi \quad (f\in\mathcal{S},\ t\in I_\sigma),$$

where $d\xi=(2\pi)^{-n}d\xi$ and $\chi(\xi)\in C_0^\infty(\boldsymbol{R}^n)$ is a cutoff function such that $0\le\chi(\xi)\le 1$, $\chi(\xi)=0$ if $|\xi|\le 1/2$ and $\chi(\xi)=1$ if $|\xi|\ge 1$. By $H_{\phi_i(t)}$ we denote the canonical transformation generated by phase function $\phi_i(t,x,\xi)$. Then we have the following

Theorem. *Assume $S_{+p,-q}(x^0,\theta^0)\neq 0$ for some $1\le p$, $q\le m$. If $g(x)\in\mathcal{S}'$ and $(x^0,\theta^0)\in WF(g)$, then*

$$u(t,x)=\sum_{i=1}^{m}F_{\sigma i,-q}(t)g(x)\quad (t\in I_\sigma,\ \sigma=\pm 1)$$

belongs to $C^\infty([-T,T],\mathcal{S}')$ and is a solution of the equation $Pu=0$ modulo a C^∞ function such that

$$H_{\phi_p(t)}(x^0,\theta^0)\in WF(u(t,\cdot))\quad (0\le t\le T),$$
$$H_{\phi_q(t)}(x^0,\theta^0)\in WF(u(t,\cdot))\quad (-T\le t\le 0).$$

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