35. *Normal Forms of Quasihomogeneous Functions with Inner Modality Equal to Five*

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§1. **Introduction.** In [3], K. Saito introduced the invariant $s(f)$ into quasihomogeneous functions with an isolated critical point 0, which is defined as the maximal quasi-degree of generators of a monomial base of the local ring $\mathcal{O}_{C^n}/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ (this local ring is denoted by R_f). And he classified quasihomogeneous functions with $s(f)=0$ and 1.

In [1], V. I. Arnol'd introduced the invariant $m_0(f)$ into quasihomogeneous functions with an isolated critical point 0, which is called the inner modality and defined as the number of generators of a monomial base of R_f on the Newton diagram and above it. And he classified quasihomogeneous functions with $m_0(f)=0$ and 1.

In [4], we classified quasihomogeneous functions with inner modality equal to 2, 3 and 4 and studied the relations of some adjaciencies among them.

In this paper, we shall give the classification of quasihomogeneous functions with inner modality equal to 5 and some adjaciencies among them.

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§2. **Classification.** Let a formal power series $f \in C[[x_1, \cdots, x_n]]$ be quasihomogeneous of type $(1; r_1, \cdots, r_n)$, i.e. the quasidegree of each monomial of f is equal to 1. This definition is equivalent to

$$f(t^{r_1}x_1, \cdots, t^{r_n}x_n)=tf(x_1, \cdots, x_n) \qquad \text{for any } t \in C.$$

K. Saito showed in [2] that if f has an isolated critical point 0, then there exists a coordinate system $(y_1, \cdots, y_n)$ such that $f=h(y_1, \cdots, y_k)+y_{k+1}^2+\cdots+y_n^2$, where h is a quasihomogeneous polynomial of type $(1; s_1, \cdots, s_k)$ $(0<s_j<1/2,\ j=1, \cdots, k,\ s_j \in Q)$. Then we call the natural number k the corank of f and call the polynomial h the residual part of f. In what follows, we may consider a quasihomogeneous polynomial of type $(1; r_1, \cdots, r_n)$ $(0<r_j \leqq 1/2,\ r_j \in Q)$ with an isolated critical point 0.

Definition 1 (Arnol'd [1]). Let f be as above. The inner modality of f is defined by the number of generators of a monomial base of R_f with quasi-degree equal to 1 and it is denoted by $m_0(f)$.

In [1], Arnol'd showed that $m_0(f)$ is given as the number of generators of a monomial base of R_f with quasi-degree less than or equal to $d-1$, where $d=n-2\sum r_j$. Note that the inner modality of f is equal to the inner modality of its residual part.

The key of the classification by inner modality is the following

Proposition 1. *If the inner modality of f is less than or equal to 5, then the corank of f is less than or equal to 4 and the quasi-homogeneous function f has the inner modality m if and only if $\#\{(i_1, \cdots, i_k) \in N^k \mid \sum_{j=1}^k i_j r_j \leqq d-1\}=m$, where $k=\mathrm{corank}(f)$.*

Proof. We have already shown in [4] that if $m_0(f)\leqq 5$, $\mathrm{corank}(f) \leqq 4$ and if $m_0(f)\leqq 5$ and $\mathrm{corank}(f)\leqq 3$, $m_0(f)=\#\{(i_1, \cdots, i_k) \in N^k \mid \sum_{j=1}^k i_j r_j \leqq d-1\}$. So we have only to prove that if $m_0(f)\leqq 5$ and corank$=4$,

$$(*) \qquad m_0(f)=\#\left\{(i_1, \cdots, i_4) \in N^4 \,\middle|\, \sum_{j=1}^4 i_j r_j \leqq d-1\right\}$$

Case 1: $0<r_1\leqq r_2\leqq r_3\leqq r_4\leqq 1/3$. 1, x_1, x_2, x_3 and x_4 are always generators of a monomial base of R_f and quasi-degree $(x_j)=r_j\leqq 1/3 \leqq d-1$ (the last inequality follows from Saito's inequality $\sum_{j=1}^4 r_j \leqq 4/3$ (see [2])). So we have $m_0(f)\geqq 5$. We consider two cases.

(1) quasi-degree $(x_1^2)=2r_1>d-1$; it is trivial that $(*)$ holds.

(2) quasi-degree $(x_1^2)=2r_1\leqq d-1$; if $2r_1\geqq 1-r_4(\geqq 2/3)$, then $r_1 \geqq 1/3$, i.e. $r_1=r_2=r_3=r_4=1/3$. It is easily seen that the quasihomogeneous polynomial with $r_j=1/3$ ($j=1, 2, 3, 4$) satisfies the formula $(*)$. If $2r_1<1-r_4$, the monomial x_1^2 is always a generator of a monomial base of R_f since $1-r_4=$the minimal quasi-degree of the partial differential $\partial f/\partial x_j$ ($j=1, 2, 3, 4$). So we have $m_0(f)\geqq 6$ since $2r_1\leqq d-1$. It is a contradiction.

Case 2: $1/3<r_4$. Since f has an isolated critical point 0, f contains the monomial x_4^n ($n\geqq 3$) or the monomial $x_4^m x_j$ ($m\geqq 2$, $j\neq 4$). By hypothesis, the first case is impossible and in the second case, we have $m=2$. So f contains the monomial $x_4^2 x_j$ ($j\neq 4$). Hence we have $r_4+r_j =1-r_4=$the minimal quasi-degree of the partial differential $\partial f/\partial x_j$ ($j=1, 2, 3, 4$). We consider two cases.

(1) $r_4\leqq d-1$: We have $m_0(f)\geqq 5$ since quasi-degree $(x_j)\leqq r_4 \leqq d-1$. If $2r_1\geqq 1-r_4$, then $2r_1\geqq r_4+r_1$ and $r_1=r_4$. It is a contradiction. If $2r_1<1-r_4$, then the monomial x_1^2 is always a generator of a monomial base of R_f. By the hypothesis that $m_0(f)\leqq 5$, we have $2r_1>d-1$. So we have the formula $(*)$.

(2) $r_4>d-1$: The quasi-degree of the partial differential $\partial f/\partial x_j\geqq 1-r_4=r_4+r_j>d-1$ for $j=1, 2, 3, 4$. If quasi-degree $(e)\leqq d-1$ for a monomial e, then the monomial e is always a generator of a monomial base of R_f. So we have the formula $(*)$.

These complete the proof of Proposition. Q.E.D.

Remark. It is easily seen that the condition $\sum r_j\geqq(2n-3)/4$ is omitted in Proposition 3.2 [4].

By making use of Proposition 1, we can carry on with the classification of quasihomogeneous functions with inner modality equal to 5 in the same say as in [4].

Theorem 1. *Residual parts of quasihomogeneous functions with inner modality equal to* 5 *are exhausted by the following table*:

Notation	Normal forms
E_{36}	x^3+y^{19}
E_{37}	x^3+xy^{13}
E_{38}	x^3+y^{20}
J_{34}	$x^3+tx^2y^6+y^{18},\ 4t^3+27\neq 0$
W_{27}	$x^4+tx^2y^5+y^{10},\ t^2-4\neq 0$
W_{29}	x^4+xy^8
W_{30}	x^4+y^{11}
Z_{33}	$x^3y+tx^2y^6+y^{16},\ 4t^3+27\neq 0$
Z_{35}	x^3y+y^{17}
Z_{36}	x^3y+xy^{12}
Z_{37}	x^3y+y^{18}
N_{26}	$x^5+tx^3y^3+xy^6,\ t^2-4\neq 0$
N_{28}	x^5+y^8
Q_{32}	$x^3+yz^2+tx^2y^5+xy^{10},\ t^2-4\neq 0$
Q_{34}	$x^3+yz^2+y^{16}$
Q_{35}	$x^3+yz^2+xy^{11}$
Q_{36}	$x^3+yz^2+y^{17}$
S_{26}	$x^2z+yz^2+ty^5z+y^9,\ t^2-4\neq 0$
S_{28}	$x^2z+yz^2+xy^7$
S_{29}	$x^2z+yz^2+y^{10}$
U^*_{20}	$x^3+xz^2+y^6+rx^2y^2+sy^2z^2+txy^2z,\ \Delta\neq 0$
U_{24}	$x^3+xz^2+y^7$
V^*_{23}	$x^2z+yz^3+y^6$
V^*_{24}	$x^2z+yz^3+xy^4$
V'_{22}	$x^3+yz^3+txy^2z+y^5,\ \Delta\neq 0$
V'^1_{24}	$x^3+y^4+z^5$
V'^2_{24}	$x^3+y^4z+yz^3$
O_{16}	$x^3+y^3+z^3+w^3+(px+qy+sz+tw)^3,\ \Delta\neq 0$
O_{20}	$x^4+y^3+z^3+w^2x+ty^2z,\ 4t^3+27\neq 0$
O_{21}	$x^2y+y^2z+xw^2+z^4$
O_{22}	$x^3+yz^2+zw^2+y^4$

Remark. In the above theorem, Δ is a polynomial of coefficients of a normal form and $\Delta\neq 0$ is the condition in which a normal form has an isolated critical point 0.

§ 3. Some adjaciencies. In this section, we give some adjaciencies among quasihomogeneous functions classified in § 2. In what follows, we denote by the notation (i.e. light-face letters E, W, Z etc. with various suffixes) of the normal form f in § 2, the family of functions $f+\sum t_j e_j$, $t \in \boldsymbol{C}$, where e_j's are generators of a monomial base of R_f with quasi-degree greater than 1.

Definition 2. Let K, L be families of functions as above. K is adjacent to L if $\overline{\bigcup_{f\in L} \mathrm{Orb}(f)} \supset \bigcup_{g\in K} \mathrm{Orb}(g)$, where Orb( ) is the orbit in the space of germs of holomorphic functions preserving 0 by the action of the group of germs of biholomorphic mapping preserving 0. We denote this adjaciencies by $L \leftarrow K$.

Definition 3. A quasihomogeneous function f is a boundary of the family of quasihomogeneous functions with inner modality equal to k if $m_0(f)$ is equal to $k+1$ and the number of generators of a monomial base of R_f with quasi-degree greater than 1 is less than $k+1$.

The relations of some adjaciencies among quasihomogeneous functions in § 2 are the following.

$$\begin{array}{l}
J_{34} \leftarrow E_{36} \leftarrow E_{37} \leftarrow E_{38} \\
W_{27} \leftarrow W_{29} \leftarrow W_{30} \\
Z_{33} \leftarrow Z_{35} \leftarrow Z_{36} \leftarrow Z_{37} \\
N_{26} \leftarrow N_{28} \\
Q_{32} \leftarrow Q_{34} \leftarrow Q_{35} \leftarrow Q_{36} \\
S_{26} \leftarrow S_{28} \leftarrow S_{29} \\
U^*_{20} \leftarrow U_{24} \\
\quad \nwarrow \\
\qquad V^*_{23} \leftarrow V^*_{24} \\
V'_{22} \leftarrow V'^{1}_{24} \\
\quad \nwarrow \\
\qquad V'^{2}_{24} \\
O_{16} \leftarrow O_{20} \leftarrow O_{21} \leftarrow O_{22}
\end{array}$$

Hence we can extend Theorem 4.2 in [4] and have the following

Theorem 2. *For* $k=0, 1, 2, 3$ *and* 4, *each family of quasihomogeneous functions with inner modality equal to* $k+1$ *is adjacent to the boundary of quasihomogeneous functions with inner modality equal to* k.

References

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