34. *An Asymptotic Property of a Certain Brownian Motion Expectation for Large Time*

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1. Let $(X(t): t\geqq 0, P)$ be the Brownian motion in $\boldsymbol{R}^d$ starting from $X(0)=0$. We give an asymptotic formula for the quantity

$$(1)\quad J(t)=J(t;\varphi)=E\left[\exp\left\{-\nu\int_{R^d}\left\{1-\exp\left(-\int_0^t\varphi(X(\sigma)-y)d\sigma\right)\right\}dy\right\}\right]$$

as $t\to\infty$, where E denotes the expectation with respect to P, φ a non-negative Borel function on $\boldsymbol{R}^d$ and $\nu>0$ a constant. Asymptotic behavior of $J(t)$ has been investigated in connection with the study of the spectral distributions of the Schrödinger operators $-1/2\Delta+q(x)$ with random potentials of the form $q(x)=\sum\varphi(x-\xi_n)$, where $\{\xi_n\}$ is the support of the Poisson random measure with intensity $\nu>0$ (see [2]–[7]).

Donsker and Varadhan [2] proved that if $\varphi(x)=o(1/|x|^{d+2})(|x|\to\infty)$ and $\int\varphi(x)dx>0$, then

$$(2)\quad \lim_{t\to\infty}t^{-d/(d+2)}\log J(t)=-k(\nu)$$

exists and

$$(3)\quad k(\nu)=\nu^{2/(d+2)}\frac{d+2}{2}(2\lambda_1/d)^{d/(d+2)},$$

where λ_1 is the smallest eigenvalue for $-1/2\Delta$ in a sphere of unit volume with zero boundary condition. On the other hand, Pastur [7] proved that if $\varphi(x)\sim K/|x|^{d+\beta}(|x|\to\infty)$, where $K>0$ and $0<\beta<2$, then

$$(4)\quad \lim_{t\to\infty}t^{-d/(d+\beta)}\log J(t)=-\kappa(\nu,\beta,K)$$

exists and

$$(5)\quad \kappa(\nu,\beta,K)=\nu K^{d/(d+\beta)}\Gamma\left(\frac{\beta}{d+\beta}\right)\Omega_d,$$

where Ω_d is the volume of a sphere of unit radius. The following theorem covers the critical case of $\varphi(x)\sim K/|x|^{d+2}(|x|\to\infty)$.

Theorem 1. *Let $(X(t), t\geqq 0)$ be the d-dimensional Brownian motion with $X(0)=0$. Suppose φ is a non-negative bounded Borel function of $\boldsymbol{R}^d$ such that $\varphi(x)\sim K/|x|^{d+2}(|x|\to\infty)$, where $K>0$. Define $J(t)$ by (1). Then for any $\nu>0$*

$$(6)\quad \lim_{t\to\infty}t^{-d/(d+2)}\log J(t)=-C(\nu,K)$$

exists and $C(\nu,K)=\inf_{f\in\mathcal{F}_0}[I(f)+\Phi(f)]$, where

(7)
$$I(f)=\frac{1}{2}\int_{R^d}|\nabla\sqrt{f}|^2\,dx,$$
$$\Phi(f)=\nu\int_{R^d}\left\{1-\exp\left(-K\int_{R^d}\frac{f(x)dx}{|x-y|^{d+2}}\right)\right\}dy$$

and $\mathcal{F}_0=\{f\in\mathcal{F}\,;\ f$ *has a bounded support and* $I(f)<\infty\}$.

Here $\mathcal{F}$ denotes the set of all probability density functions on $\boldsymbol{R}^d$ and ∇ denotes the usual gradient vector in the distribution sense.

Remarks. (i) Theorem 1 is still valid if $K/|x|^{d+2}$ is replaced by any $\omega(x)>0$ which is homogeneous of degree $-(d+2)$, i.e., $\omega(\lambda x)=\omega(x)/\lambda^{d+2}$, $\lambda>0$, and continuous in $x\neq 0$.

(ii) Furthermore, if the Brownian motion is replaced by a d-dimensional symmetric stable process of index α $(0<\alpha<2)$, then Theorem 1 holds with $d+2$ and $I(f)$ replaced by $d+\alpha$ and $I^{(\alpha)}(f)=2^{-1}\int dx\int|\sqrt{f(x+y)}-\sqrt{f(x)}|^2 n(dy)$, respectively, where $n(dy)$ is the Lévy measure of the stable process.

We next give some information as to how $C(\nu, K)$ in Theorem 1 depends on K and ν and how it is related to $k(\nu)$ in (3) and $\kappa(\nu, 2, K)$ in (5). In the following we write $\kappa(\nu, K)$ for $\kappa(\nu, 2, K)$.

Theorem 2. (i) *$C(\nu, K)$ is strictly increasing, concave and continuous both in $K>0$ and in $\nu>0$.*

(ii) $C(\nu, K)>\max\{k(\nu), \kappa(\nu, K)\}$.

(iii) *$C(\nu, K)\downarrow k(\nu)$ as $K\downarrow 0$ and $C(\nu, K)\sim\kappa(\nu, K)$ as $K\uparrow\infty$.*

(iv) *$C(\nu, K)\sim k(\nu)$ as $\nu\downarrow 0$ and $C(\nu, K)\sim\kappa(\nu, K)$ as $\nu\uparrow\infty$.*

The proof of Theorem 1 will be given in §§ 2 and 3 and the proof of Theorem 2 will be given in § 4.

2. Proof of Theorem 1 (upper bound). In this section we prove

(8)
$$\overline{\lim_{t\to\infty}}\, t^{-d/(d+2)}\log J(t\,;\varphi)\leqq -C(\nu, K).$$

Let $\psi=\rho*\varphi$ (convolution), where $\rho\in\mathcal{F}$. We first prove

(9)
$$\overline{\lim_{t\to\infty}}\, t^{-d/(d+2)}\log J(t\,;\psi)\leqq -C(\nu, K).$$

To prove (9) we will use the argument similar to that of the upper bound in [2]. In particular, we appeal to the Donsker-Varadhan large deviation theorem for the Brownian motion on a torus ([1]). Let $M>0$ be given. Let $\boldsymbol{T}$ be a d-dimensional torus of size M and let $G=\{(Mn_1, \cdots, Mn_d): n_i\in\boldsymbol{Z}, i=1, \cdots, d\}$ so that $\boldsymbol{T}=\boldsymbol{R}^d/G$. We think of $\boldsymbol{T}$ as $[0, M]^d\subset\boldsymbol{R}^d$ with the sides identified. Let $\mathcal{F}_M$ be the set of all probability density functions on $\boldsymbol{T}$, but periodically extended to the whole space $\boldsymbol{R}^d$. For $g\in\mathcal{F}_M$ let

$$\Phi^M(g)=\nu\int_T\left\{1-\exp\left(-K\int_{R^d}\frac{g(x)dx}{|x-y|^{d+2}}\right)\right\}dy \text{ and } I^M(g)=\frac{1}{2}\int_T|\nabla\sqrt{g}|^2dx$$

if the right hand side makes sense, otherwise $I^M(g)=\infty$.

Lemma 2.1. *Let* $\psi(x)=\rho*\varphi(x)\equiv\int_{R^d}\rho(x-y)\varphi(y)dy$ *with* $\rho\in\mathcal{F}$. *Then*

(10) $$\overline{\lim_{t\to\infty}}\, t^{-d/(d+2)}\log J(t;\psi)\leqq-\inf_{g\in\mathcal{F}_M}[I^M(g)+\Phi^M(g)].$$

Proof. For any $\varepsilon>0$ define $k_\varepsilon\in\mathcal{F}_M$ by $k_\varepsilon(x)=\sum_{\gamma\in G}\rho_\varepsilon(x+\gamma)$, $x\in R^d$, where $\rho_\varepsilon(x)=\varepsilon^{-d}\rho(\varepsilon^{-1}x)$. Moreover, for any trajectory $\omega=x(\cdot)$ on T and any $\tau>0$ define $g_\tau(\omega,\cdot)\in\mathcal{F}_M$ by $g_\tau(\omega,y)=\tau^{-1}\int_0^\tau k_{\varepsilon(\tau)}(x(\sigma)-y)d\sigma$, $y\in R^d$, where $\varepsilon(\tau)=\tau^{-1/d}$. Let $X(t)$, $t\geqq 0$ be a trajectory in R^d with $X(0)=0$. Define, for each $s>0$, a new trajectory $X^s(\cdot)$ by $X^s(t)=s^{-1}X(s^2t)$. Let $\pi: R^d\to T$ be the canonical projection. Set $\tau=\tau(t)=t^{d/(d+2)}$ and $s=s(\tau)=\tau^{1/d}(=t^{1/(d+2)})$. By change of variables and using the argument in [2, p. 562], we have for the given $\psi=\rho*\varphi$

$$\exp\left\{-\nu\int_{R^d}\left\{1-\exp\left(-\int_0^t\psi(X(\sigma)-y)\,d\sigma\right)\right\}dy\right\}$$
$$\leqq\exp\{-\tau\Phi_\tau^M(g_\tau(\omega,\cdot))\}(\omega=\pi(X^s(\cdot))),$$

where $\Phi_\tau^M(g)=\nu\int_T\left\{1-\exp\left(-\int_{R^d}\varphi_\tau(x-y)g(x)dx\right)\right\}dy$, $g\in\mathcal{F}_M$ and $\varphi_\tau(z)=\tau^{(d+2)/d}\varphi(\tau^{1/d}z)$. Since the laws of $X^s(\cdot)$ and $X(\cdot)$ are identical, we have

(11) $$J(t;\psi)\leqq E[\exp\{-\tau\Phi_\tau^M(g_\tau(\pi(X(\cdot)),\cdot))\}]\ (\tau=t^{d/(d+2)}).$$

Since $\varphi_\tau(z)\to K/|z|^{d+2}$ as $\tau\to\infty$, we can check by using Fatou's lemma twice that if $g^\tau, g\in\mathcal{F}_M$ satisfy $g^\tau\to g$ in $L^1(T,dx)$ as $\tau\to\infty$, then $\underline{\lim}\,\Phi_\tau^M(g^\tau)\geqq\Phi^M(g)$. Thus it follows from Corollary to Theorem 5.1 of [1] that

(12) $$\overline{\lim_{\tau\to\infty}}\frac{1}{\tau}\log E[\exp\{-\tau\Phi_\tau^M(g_\tau(\pi(X(\cdot)),\cdot))\}]\leqq-\inf_{g\in\mathcal{F}_M}[I^M(g)+\Phi^M(g)].$$

Combining (12) with (11), we have (10). Q.E.D.

To establish (9) we have only to prove the following lemma since $M>0$ is arbitrary.

Lemma 2.2. $\sup_{M>0}\inf_{g\in\mathcal{F}_M}[I^M(g)+\Phi^M(g)]\geqq C(\nu,K)$.

This is the analogue of Lemma 3.5 of [2] and can be proved similarly with a slight modification. We omit the proof.

The following lemma reduces (8) to (9).

Lemma 2.3. *For each* $0<a<1$ *there is a* $\tilde\varphi(x)\geqq 0$ *with the property that* $\tilde\varphi(x)\sim aK/|x|^{d+2}(|x|\to\infty)$ *and a* $\rho\in\mathcal{F}$ *such that* $\tilde\psi(x)\equiv\rho*\tilde\varphi(x)\leqq\varphi(x)$ *for all* $x\in R^d$.

Proof. Let $\tilde\varphi(x)=aK/|x|^{d+2}$ if $|x|\geqq R$, $\tilde\varphi(x)=0$ otherwise and let $\rho\in\mathcal{F}$ satisfy $\{\rho>0\}\subset\{|x|<\delta\}$. Then one can check that $\rho*\tilde\varphi\leqq\varphi$ for large $R>0$ and small $\delta>0$. Q.E.D.

It follows from (9) that for $\tilde\psi$ in Lemma 2.3

$$\overline{\lim_{t\to\infty}}\, t^{-d/(d+2)}\log J(t;\tilde\psi)\leqq -C(\nu,aK).$$

Since $J(t;\varphi)\leqq J(t;\tilde\psi)$, we have (8) with aK replacing K. Letting $a\uparrow 1$, we have (8) by Theorem 2 (i).

3. Proof of Theorem 1 (lower bound). In this section we prove

(13) $$\varliminf_{t\to\infty} t^{-d/(d+2)} \log J(t) \geqq -C(\nu, K).$$

By the inequality due to Pastur [7] (see also [4], [5]) we have

(14) $$J(t) \geqq \|\sqrt{f}\|_\infty^{-1} \|\sqrt{f}\|_1^{-1} \exp\{-[tI(f)+\Psi_t(f)]\}, \qquad f\in\mathcal{F}_0,$$

where $\Psi_t(f)=\nu\int\left\{1-\exp\left(-t\int\varphi(x-y)f(x)dx\right)\right\}dy$, $\|u\|_\infty=\text{ess. sup }|u|$ and $\|u\|_1=\int|u|dx$. Define $f_t\in\mathcal{F}_0$, $t>0$ by $f_t(x)=t^{-d/(d+2)}f(t^{-1/(d+2)}x)$ for any bounded $f\in\mathcal{F}_0$ and substitute f_t for f in (14). Then, by change of variables, we have

$$J(t) \geqq \|\sqrt{f}\|_\infty^{-1} \|\sqrt{f}\|_1^{-1} \exp\{-t^{d/(d+2)}[I(f)+\Phi_t(f)]\},$$

where $\Phi_t(f)=\nu\int\left\{1-\exp\left(-\int t\varphi(t^{1/(d+2)}(x-y))f(x)dx\right)\right\}dy$, and hence

$$\varliminf_{t\to\infty} t^{-d/(d+2)} \log J(t) \geqq -[I(f)+\varlimsup_{t\to\infty}\Phi_t(f)].$$

Note that there is an $A>0$ such that $\varphi(x)\leqq A/|x|^{d+2}$, $x\in \boldsymbol{R}^d$ since φ is bounded. Thus we can prove $\varlimsup \Phi_t(f)\leqq\Phi(f)$ by using the Lebesgue-Fatou theorem twice. Hence we have

(15) $$\varliminf_{t\to\infty} t^{-d/(d+2)} \log J(t) \geqq -[I(f)+\Phi(f)]$$

for any bounded $f\in\mathcal{F}_0$. It is easy to see by a truncation argument that (15) holds for any $f\in\mathcal{F}_0$, proving (13).

4. Proof of Theorem 2. The first assertion follows from the definition of $C(\nu, K)$ and (iii), (iv). Define $\Phi(f\,;a)$ for $a>0$ by $\Phi(f)$ in (7) with aK replacing K and define $\tilde{C}(a,\nu,K)=\inf[aI(f)+\Phi(f)]\,(f\in\mathcal{F}_0)$ for $a>0$. Noting that $I(f_R)=R^{-2}I(f)$, $\Phi(f_R)=R^d\Phi(f\,;1/R^{d+2})$, where $f_R(x)=R^{-d}f(R^{-1}x)$, $R>0$, we have

(16) $$C(\nu,K)=\nu^{2/(d+2)}C(1,\nu K)=\nu K^{d/(d+2)}\tilde{C}(\nu^{-1}K^{-1},1,1).$$

Thus Theorem 2 follows from the following

Lemma 4.1. (i) *$C(1,K)\downarrow\inf[I(f)+|\{f>0\}|]=k(1)$ as $K\downarrow 0$.*

(ii) *$\tilde{C}(a,1,1)\downarrow\int\{1-\exp(-1/|y|^{d+2})\}dy=\kappa(1,1)$ as $a\downarrow 0$.*

Here $|A|$ denotes the Lebesgue measure of the set A.

Proof. Equalities in (i) and (ii) are known ([2], [7]). Since $C(1,K)\downarrow\inf[I(f)+|\{U=\infty\}|]$ $(K\downarrow 0)$, where $U(y)=\int\frac{f(x)dx}{|x-y|^{d+2}}$, (i) is a consequence of the fact that $U(y)=\infty$ if and only if $f(y)>0$ a.e. for each $f\in\mathcal{F}_0$. "If" part follows from $U(y)\geqq r^{-d-2}\int_{|x|<r}f(x+y)dx$ since $r^{-d}\int_{|x|<r}f(x+y)dx\to\text{const.}\times f(y)$ a.e. $(r\downarrow 0)$ by the Lebesgue theorem (see [8, I]). "Only if" part follows from

$$\int_{\{f(y)=0\}}U(y)dy\leqq\int dy\int\frac{|\sqrt{f(y+x)}+\sqrt{f(y-x)}-2\sqrt{f(y)}|^2}{|x|^{d+2}}dx$$
$$=\text{const.}\times I(f)<\infty$$

(see [8, VIII 5.2] for the equality). It is easy to see that $\tilde{C}(a, 1, 1) \downarrow \inf \Phi(f)$ $(a \downarrow 0)$ with $\nu = K = 1$. We have $\Phi(f) \geqq \kappa(1, 1)$ by Jensen's inequality, while we can choose $f_\varepsilon \in \mathscr{F}_0$ such that $\Phi(f_\varepsilon) \to \kappa(1, 1)$, proving (ii). Q.E.D.

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