33. *On Stochastic Differential Equations Characterizing some Singular Diffusion Processes*

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1. **Introduction.** Let $a(x)=\{a_{ij}(x)\,;\,1\leqq i,\,j\leqq d\}$ and $\alpha(x)=\{\alpha_{ij}(x)\,;\,2\leqq i,\,j\leqq d\}$ be two systems of nonnegative definite $C_b^2(R^d)$ functions such that $a_{11}(x)\geqq c$ for some positive constant c, where $C_b^k(R^d)$ denotes the class of all functions with bounded continuous derivatives up to the k-th order. Let μ be a bounded measure on R^1 singular with respect to Lebesgue measure and $\mathcal{E}(f,g)$, the symmetric form defined by

$$(1)\qquad \mathcal{E}(f,g)=\frac{1}{2}\sum_{i,j=1}^{d}\int_{R^d} a_{ij}(x)\partial_i f(x)\partial_j g(x)dx + \frac{1}{2}\sum_{i,j=2}^{d}\int_{R^d}\alpha_{ij}(x)\partial_i f(x)\partial_j g(x)\,\eta(dx)$$

for $f,\,g\in C_0^\infty(R^d)$, where $\partial_i=\partial/\partial x_i$, $\eta(dx)=\mu(dx_1)dx_2\cdots dx_d$ and $C_0^\infty(R^d)$ is the class of all $C^\infty(R^d)$ functions with compact support.

The purpose of this paper is to describe the stochastic differential equation (SDE in abbreviation) characterizing the minimal diffusion process $(X^0(t),P_x^0)$ associated with the symmetric form $(\mathcal{E},C_0^\infty(R^d))$. All the proofs are only sketched or omitted here, they will be published elsewhere.

2. **Existence of $(X^0(t),P_x^0)$.** Let $\nu(dx)=dx+\eta(dx)$ and let Λ be a Borel set in R^1 with full Lebesgue measure satisfying $\mu(\Lambda)=0$. Set $\Gamma=R^1-\Lambda$. Then the form $\mathcal{E}$, considered as a form on $L^2(d\nu)$, is expressed as

$$\mathcal{E}(f,g)=-\frac{1}{2}\int f(x)\Big\{I_\Lambda(x_1)\sum_{i,j=1}^{d}\partial_i(a_{ij}\partial_j g)(x) + I_\Gamma(x_1)\sum_{i,j=2}^{d}\partial_i(\alpha_{ij}\partial_j g)(x)\Big\}\nu(dx)$$

for $f,\,g\in C_0^\infty(R^d)$. Hence it is closable on $L^2(d\nu)$ (see [1, Problem 1.1.2]). Therefore, by the results of Fukushima [1, Theorems 2.1.1 and 2.1.2], its smallest closed extension is a regular Dirichlet form on $L^2(d\nu)$ with local property. By another result of Fukushima [1, Chapter 6], there exists a ν-symmetric diffusion process associated with the smallest closed extension. Changing the time, we have the following

Theorem 1. *There exists a symmetric* (with respect to Lebesgue measure) *diffusion process $(X^0(t),P_x^0)$ the Dirichlet form of which is the smallest closed extension of $(\mathcal{E},C_0^\infty(R^d))$ on $L^2(dx)$.*

If $a=\alpha$ and if a is locally uniformly positive definite and belongs to $C^{[(d-1)/2]+1}(R^d)$, this result was proved by Tomisaki [4] by an analytic method. Moreover she discussed the regularity properties of the resolvent. The problem constructing probabilistically singular diffusion processes such as $(X^0(t), P_x^0)$ was first treated by Ikeda and Watanabe [2].

3. The SDE determining the diffusion process $(X^0(t), P_x^0)$. We shall start with a slightly more general setting. Let a and α be those given in §1. There exist $C_b^1(R^d)$ functions $\sigma(x)=\{\sigma_{ij}(x)\,;\,1\leqq i,\,j\leqq d\}$ and $\tau(x)=\{\tau_{ij}(x)\,;\,2\leqq i,\,j\leqq d\}$ such that $a=\sigma\cdot{}^t\sigma$ and $\alpha/a_{11}=\tau\cdot{}^t\tau$. Transforming σ by an orthogonal matrix, we can always assume that $\sigma_{1j}(x)=\sqrt{a_{11}(x)}\delta_{1j}$. For arbitrary $C_b^1(R^d)$ functions $b(x)=\{b_i(x)\,;\,1\leqq i\leqq d\}$ and $\beta(x)=\{\beta_i(x)\,;\,2\leqq i\leqq d\}$, consider the following SDE.

$$(2)\qquad \begin{aligned} dX_i(t)&=\sum_{j=1}^{d}\sigma_{ij}(X(t))dB_j(t)+b_i(X(t))dt\\ &\quad+\sum_{j=1}^{d}\tau_{ij}(X(t))dM_j(t)+\beta_i(X(t))l_\mu(dt)\end{aligned}$$

$(i=1,\cdots,d)$, where we set $\tau_{1k}=\tau_{k1}=\beta_1=0$ for all $k=1,\cdots,d$.

Similar to S. Watanabe [5], the solution of the SDE (2) is defined as follows.

Definition. The solution of the SDE (2) is a system of stochastic processes $\{X(t), B(t), M(t), L(t)\}$ on a probability space $(\Omega, \mathcal{F}_t, P)$ which satisfies the following conditions.

(i) They are $\mathcal{F}_t$-adapted.

(ii) $B(t)=(B_i(t))_{i=1,\dots,d}$ is a d-dimensional $\mathcal{F}_t$-Brownian motion.

(iii) $L(t)=\{l(t, x_1)\,;\,x_1\in R^1\}$ is a family of (t, x_1)-continuous non-negative increasing processes satisfying

$$\int_0^t I_{\{X_1(s)=x_1\}}l(ds, x_1)=l(t, x_1)$$

for all $x_1\in R^1$ and $t\geqq 0$, and

$$\int_0^t f(X_1(s))a_{11}(X(s))ds=\int_{R^1} l(t, x_1)f(x_1)dx_1$$

for all $t\geqq 0$ and $f\in C_0^\infty(R^1)$.

(iv) $M(t)=(M_i(t))_{i=2,\dots,d}$ is a family of continuous $\mathcal{F}_t$-local martingales satisfying $\langle M_i, M_j\rangle(t)=\delta_{ij}l_\mu(t)$ and $\langle B_i, M_j\rangle(t)=0$, where

$$l_\mu(t)=\int l(t, x_1)\mu(dx_1).$$

(v) $X=(X, B, M, L)$ satisfies (2).

If the distribution of $X(t)$ is uniquely determined by that of $X(0)$, then we shall say that the uniqueness of the solution of (2) holds.

Similar to [5], for any probability measure ξ on R^d, there exists a unique solution of (2) having ξ as its initial distribution. Denote P_x the measure which defines the solution of (2) with initial condition

$X(0)=x$, then we have the following two lemmas.

Lemma 1. *For all $f,\ g\in C_0^\infty(R^d)$*

$$(3)\qquad \lim_{t\to 0}\frac{1}{t}\int f(x)E_x\Big[\int_0^t g(X(s))l_\mu(ds)\Big]dx=\int f(x)g(x)a_{11}(x)\eta(dx).$$

Roughly speaking, this result says that $l_\mu(t)$ is the continuous additive functional of $X(t)$ associated with the smooth (or Revuz) measure $\eta(dx)$ (see [1], [3]). By using Ito's formula and Lemma 1, we have

Lemma 2. *If we denote P_t the transition function of the diffusion process $(X(t), P_x)$, then, for all $f,\ g\in C_0^\infty(R^d)$,*

$$(4)\qquad \lim_{t\to 0}\frac{1}{t}\int f(x)(I-P_t)g(x)dx=(f,\ -Ag)+(f,\ -Lg)_\eta,$$

where $(\ ,\ )$ and $(\ ,\ )_\eta$ are the inner products relative to the measures dx and $\eta(dx)$ respectively, and A and L are the differential operators defined by

$$Ag(x)=\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial_i\partial_j g(x)+\sum_{i=1}^d b_i(x)\partial_i g(x),$$

$$Lg(x)=\frac{1}{2}\sum_{i,j=2}^d \alpha_{ij}(x)\partial_i\partial_j g(x)+a_{11}(x)\sum_{i=2}^d \beta_i(x)\partial_i g(x).$$

Lemma 2 implies that $C_0^\infty(R^d)$ is contained in the domain of the (not necessarily symmetric) Dirichlet form on $L^2(dx)$ associated with the diffusion process $(X(t), P_x)$ and that the form $\tilde{\mathcal{E}}$ is given by

$$\tilde{\mathcal{E}}(f,\ g)=\mathcal{E}(f,\ g)+\int f(x)\sum_{i=1}^d\Big\{b_i(x)-\frac{1}{2}\sum_{j=1}^d\partial_j a_{ji}(x)\Big\}\partial_i g(x)dx$$
$$+\int f(x)\sum_{i=2}^d\Big\{a_{11}(x)\beta_i(x)-\frac{1}{2}\sum_{j=2}^d\partial_j\alpha_{ji}(x)\Big\}\partial_i g(x)\eta(dx).$$

In particular, if we take

$$(5)\qquad b_i(x)=\frac{1}{2}\sum_{j=1}^d\partial_j a_{ji}(x)\quad\text{and}\quad \beta_i(x)=\frac{1}{2a_{11}(x)}\sum_{j=2}^d\partial_j\alpha_{ji}(x),$$

then the Dirichlet form of $(X(t), P_x)$ becomes an extension of $(\mathcal{E}, C_0^\infty(R^d))$ on $L^2(dx)$.

Conversely, suppose that the process $X^0(t)$ in § 2 is a process on (Ω^0, P_x^0). A probability space $(\tilde{\Omega}^0, \tilde{P}_x^0)$ is called an enlargement of (Ω^0, P_x^0) if there exists a mapping ϕ of $\tilde{\Omega}^0$ onto Ω^0 such that $P_x^0=\tilde{P}_x^0\circ\phi^{-1}$. In this case we write $X^0(t)$ for $X^0(t, \phi\circ\tilde{\omega})$. We can show that, for q.e. x, there exists an enlargement $(\tilde{\Omega}^0, \tilde{P}_x^0)$ of (Ω^0, P_x^0) and stochastic processes $(B^0(t), M^0(t), L^0(t))$ over $(\tilde{\Omega}^0, \tilde{P}_x^0)$ such that (X^0, B^0, M^0, L^0) is a solution of (2).

Thus we have the following

Theorem 2. *Suppose that the coefficients b and β of* SDE (2) *are those given by* (5). *Then the diffusion process $(X(t), P_x)$ is equivalent to $(X^0(t), P_x^0)$, where the equivalence means that the distributions of*

$X(t)$ under P_x and $X^0(t)$ under P_x^0 are the same for every x outside a set of properly exceptional set (see [1, § 4.3]).

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References

[1] M. Fukushima: Dirichlet Forms and Markov Processes. Kodansha Scientific, Tokyo (1980).
[2] N. Ikeda and S. Watanabe: The local structure of a class of diffusions and related problems. Proc. 2nd Japan-USSR symp. on probability theory. Lecture Notes in Math., vol. 330, Springer Verlag, Berlin-Heidelberg-New York, pp. 124–169 (1973).
[3] D. Revuz: Mesures associées aux fonctionelles de Markov I. Trans. Amer. Math. Soc., **148**, 501–531 (1970).
[4] M. Tomisaki: A construction of diffusion processes with singular product measures. Z. Wahrscheinlichkeitstheorie verw. Gebiete, **53**, 51–70 (1980).
[5] S. Watanabe: On stochastic differential equations for multidimensional diffusion processes with boundary conditions I. J. Math. Kyoto Univ., **11**, 169–180 (1971); ditto. II. ibid., **11**, 545–551 (1971).