32. *An Approximate Positive Part of Essentially Self-Adjoint Pseudo-Differential Operators. II*

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§ 1. **Introduction.** Let $a(x, \xi)$ be a real valued symbol function belonging to the class $S^1_{10}(\mathrm{R}^n)$ of Hörmander [2], that is, for any pair of multi-indices α and β, we have

$$\sup (1+|\xi|^2)^{(|\beta|-1)/2}|D^\alpha_x D^\beta_\xi a(x, \xi)|<\infty,$$

where we used usual multi-index notation. As the continuation of the previous note [1], we treat the Weyl quantization $a^w(x, D)$ of it, which is defined as

$$(1.1) \qquad a^w(x, D)u(x)=\left(\frac{1}{2\pi}\right)^n \int_{\mathrm{R}^n}\int_{\mathrm{R}^n} a\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\cdot\xi}u(y)\, dy\, d\xi.$$

Cf. Weyl [6], Voros [5], and Hörmander [3].

Let (,) and $\|\ \|$ denote the inner product and the norm, respectively, in $L^2(\mathrm{R}^n)$. In the previous note, we reported the following

Theorem 1. *Let ε be an arbitrary small positive number. Then, using the symbol function $a(x, \xi)$, we can construct three bounded linear operators π^+, π^- and R in $L^2(\mathrm{R}^n)$ with the following properties:*

1) *Both π^+ and π^- are non-negative symmetric operators.*

2) *There exists a positive constant C such that we have*

$$(1.2) \qquad Re(\pi^+ a^w(x, D)u, u)\geq -C\|u\|^2$$

$$(1.3) \qquad -Re(\pi^- a^w(x, D)u, u)\geq -C\|u\|^2$$

for any $u\in \mathcal{S}(\mathrm{R}^n)$.

3) $\pi^+ + \pi^- = I+R, \qquad \|R\|<\varepsilon, \qquad$ *and*

$$\|a^w(x, D)R\|<\infty, \qquad \|R\, a^w(x, D)\|<\infty.$$

Let

$$\mathfrak{C}^+(a)=\{(x, \xi)\,|\,a(x, \xi)\geq 0\}$$

and

$$\mathfrak{C}^-(a)=\{(x, \xi)\,|\,a(x, \xi)\leq 0\}.$$

We call $\mathfrak{C}^0(a)=\mathfrak{C}^+(a)\cap\mathfrak{C}^-(a)$ the characteristic set of a. The aim of this note is to show the following

Theorem 2. *Let $a(x, \xi)$ and $p(x, \xi)$ be two real valued functions in $S^1_{10}(\mathrm{R}^n)$. Suppose the following two conditions hold:*

(A) $\mathfrak{C}^+(a)\subset\mathfrak{C}^+(p), \qquad \mathfrak{C}^-(a)\subset\mathfrak{C}^-(p).$

(B) *There exists a positive constant C such that*

$$(1.4) \qquad |\mathrm{grad}_x p(x, \xi)|\leq C|\mathrm{grad}_x a(x, \xi)|$$

$$(1.5) \qquad |\mathrm{grad}_\xi p(x, \xi)|\leq C|\mathrm{grad}_\xi a(x, \xi)|$$

at every $(x, \xi) \in \mathfrak{C}^0(a)$. *Let* π^+, π^- *and* R *be the linear operators constructed for* $a^w(x, D)$ *in Theorem* 1. *Then we have*

(1.6) $\qquad Re(\pi^+ p^w(x, D)u, u) \geq -C\|u\|^2$

(1.7) $\qquad -Re(\pi^- p^w(x, D)u, u) \geq -C\|u\|^2 \qquad$ *for any* $u \in \mathcal{S}(\mathbf{R}^n)$

and

$$\|R\, p^w(x, D)\| < \infty, \qquad \|p^w(x, D)R\| < \infty$$

with some positive constant C.

§2. Sketch of the proof of Theorem 2. Let $\{Q_\nu\}_{\nu=1}^{\infty}$ be the partition of $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$ into closed rectangles $Q_\nu = Q_{\nu x} \times Q_{\nu \xi}$ in [1]. Let δ_μ = diam. of $Q_{\mu x}$ and ε_μ = diam. of $Q_{\mu\xi}$. Let $\varphi_\nu(x, \xi)$ and $\psi_\nu(x, \xi)$ be functions as in [1]. At every point $w = (x, \xi)$ in the interior of the rectangle Q_μ, we assign the quadratic form

$$g_w : \mathbf{R}_x^n \times \mathbf{R}_\xi^n \ni (t, \tau) \longrightarrow g_w(t, \tau) = \delta_\mu^{-2}|t|^2 + \varepsilon_\mu^{-2}|\tau|^2.$$

The correspondence $w \to g_w$ is a discontinuous σ-temperate Riemannian metric in the sense of Hörmander [3]. This metric g_w is equivalent to the metric g_w in [1]. Following [3], we define

(2.1) $$g_w^\sigma(t, \tau) = \varepsilon_\mu^2 |t|^2 + \delta_\mu^2 |\tau|^2$$

and

(2.2) $$h(w) = \delta_\mu^{-1} \varepsilon_\mu^{-1}$$

if $w = (x, \xi)$ is an interior point of Q_μ. We showed in [1] that $a \in S(h^{-1}, g)$ and both sets $\{\varphi_\mu\}$, $\{\psi_\mu\}$ are bounded in $S(1, g)$. (See Hörmander [3] for the definition of the class $S(h^{-1}, g)$ and $S(1, g)$.)

We can prove

Proposition 1. *Under the assumptions* (A) *and* (B), *the function* $p(x, \xi)$ *belongs to the class* $S(h^{-1}, g)$, i.e., *for any multi-indices* α *and* β, *we have the estimate*

(2.3) $$|D_x^\alpha D_\xi^\beta\, p(x, \xi)| \leq C_{\alpha\beta}\, \delta_\mu^{1-|\alpha|} \varepsilon_\mu^{1-|\beta|}$$

if $(x, \xi) \in 4Q_\mu$.

Corresponding to Lemma 2.1 of [1], we can prove

Lemma 2. *Let* $h_\mu = \delta_\mu^{-1} \varepsilon_\mu^{-1}$. *Let* π_μ^+, π_μ^-, R_μ *and* ϕ_μ *be as in Lemma* 2.1 *of* [1]. *Then,*

(i) *There exists a positive constant* C *such that we have*

(2.4) $\qquad Re(\pi_\mu^+ p \psi_\mu^w(x, D) \varphi_\mu^w(x, D)u, \varphi_\mu^w(x, D)u) \geq -C N^2 \|\varphi_\mu^w(x, D)u\|^2,$

(2.5) $\qquad -Re(\pi_\mu^- p \psi_\mu^w(x, D) \varphi_\mu^w(x, D)u, \varphi_\mu^w(x, D)u) \geq -C N^2 \|\varphi_\mu^w(x, D)u\|^2,$

for any u *in* $\mathcal{S}(\mathbf{R}^n)$.

Sketch of the proof of Lemma 2. In the case (I) of Lemma 1.2 of [1], we have

(2.6) $\qquad |p(x, \xi)| \leq C N^2 \qquad$ for any $(x, \xi) \in 4Q_\mu$,

because of assumption (A) and Proposition 1. This proves (2.4) and (2.5). In the case (II) of Lemma 1.2 of [1], we have

$$p(x, \xi) \geq 0 \qquad \text{for any } (x, \xi) \in 4Q_\mu$$

because of assumption (A). Hence (2.4) and (2.5) hold in this case. Case (III) of Lemma 1.2 of [1] can be treated in the similar manner.

Lemma 3. *If case* $(\mathrm{IV})_k$ *of Lemma* 2.1 *of* [1] *holds, then there exists a non-negative function* $q(x, \xi)$ *of* $(x, \xi) \in 4Q_\mu$ *such that*

(2.7) $$p(x, \xi) = q(x, \xi)\, a(x, \xi) \qquad \text{for any } (x, \xi) \in 4Q_\mu.$$

For any multi-indices α *and* β, *we have*

(2.8) $$|D_x^\alpha D_\xi^\beta q(x, \xi)| \le C_{\alpha\beta}\, \delta_\mu^{-|\alpha|} \varepsilon_\mu^{-|\beta|} \qquad \text{for } (x, \xi) \in 4Q_\mu.$$

Let $\chi_\mu(x, \xi) = \phi_\mu(x, \xi)^{1/2}$, which we may assume of class C^∞. The function $q\chi_\mu$ belongs to $S(1, g)$. We define the operator $(q\chi_\mu)^w(x, D)$ and we have

$$(p\phi_\mu)^w(x, D) = (q\chi_\mu)^w(x, D)(a\chi_\mu)^w(x, D) + r^w(x, D)$$

where $r_\mu = q\chi_\mu a\chi_\mu - (q\chi_\mu) \# (a\chi_\mu)$. Since $q\chi_\mu(x, \xi) \ge 0$, we can apply the technique of Nirenberg Trévès (cf. Lemma 3.1 of [4]). Thus, in the case $(\mathrm{IV})_k$ of Lamma 1.2, we can prove (2.4) and (2.5).

Similar discussions prove (2.4) and (2.5) in the case $(\mathrm{V})_k$ of Lemma 2.1 in [1].

Theorem 2 follows from Lemma 2 if we can prove that the operators

(2.9) $$R_1' = \sum_\mu \varphi_\mu^w(x, D)\pi_\mu^+ [\varphi_\mu^w(x, D), p\phi_\mu^w(x, D)]$$

(2.10) $$R_2' = \sum_\mu \varphi_\mu^w(x, D)\pi_\mu^+ \varphi_\mu^w(x, D)(p^w(x, D) - (p\phi_\mu)^w(x, D))$$

are bounded (cf. (3.7) and (3.8) of [1]). In order to prove the boundedness of these operators as well as estimates (3.5) and (3.9) of [1], we use the fundamental estimate of Hörmander, which is implicit in [3]. Let $p_\mu(x, \xi)$ and $p_\nu(x, \xi)$ be C^∞ functions with compact supports. For any integer $L \ge 0$ and $w = (x, \xi) \in \mathbf{R}_x^n \times \mathbf{R}_\xi^n$, we put

(2.11) $$p_{\mu\nu}^L(x, \xi) = p_\mu \# p_\nu(w) - \sum_{j<L} \frac{1}{j!}\left(\frac{i}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)^j p_\mu(x, \xi)p_\nu(y, \eta)|_{(y,\eta)=(x,\xi)}.$$

For any $w = (x, \xi) \in \mathbf{R}_x^n \times \mathbf{R}_\xi^n$, we put

$$d_\mu(w) = \inf_{w' \in (15/8)Q_\mu} g_w^\sigma(w - w').$$

Then Hörmander's estimate can be stated as follows:

Lemma 4. *Let* $p_\mu(x, \xi)$ *and* $p_\nu(x, \xi)$ *be* C^∞ *functions. Suppose that* $\operatorname{supp} p_\mu \subset (7/4)Q_\mu$ *and* $\operatorname{supp} p_\nu \subset (7/4)Q_\nu$. *Then, for any non-negative integers* k *and* l, *there exist positive constants* C, μ *and* M *such that for any* $w = (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$

(2.12) $$|p_{\mu\nu}^L|_l^g(w) \le C\, h(w)^L (1 + d_\mu(w) + d_\nu(w))^{-k} \times \sup_{j_1 + j_2 \le M}\left[\sup_{w_1} |p_\mu|_{j_1}^g(w_1)\right] \cdot \left[\sup_{w_2} |p_\nu|_{j_2}^g(w_2)\right].$$

See [3] for the definition of the seminorm $|p_\mu|_j^g(w)$.

In taking summation with respect to μ, we use

Lemma 5. *There exists a positive number* M *such that if* $k > M$

we have

(2.13) $$\sum_{\mu}(1+A+d_{\mu}(w))^{-k}<C(1+A)^{M-k}$$

for any positive number A. Here C is independent of A.

References

[1] Fujiwara, D.: An approximate positive part of essentially self-adjoint pseudo-differential operators. I. Proc. Japan Acad., **57**A, 1–6 (1981).

[2] Hörmander, L.: Pseudo-differential operators and hypo-elliptic equations. Proc. Symp. Pure Math., **10**, 118–196 (1966).

[3] ——: The Weyl calculus of pseudo-differential operators. Comm. Pure Appl. Math., **32**, 359–443 (1979).

[4] Nirenberg, L., and Trévès, F.: On local solvability of linear partial differential equations, part II, Sufficient conditions. ibid., **23**, 459–510 (1970).

[5] Voros, A.: An algebra of pseudo-differential operators and the asymptotics of quantum mechanics. J. Funct. Anal., **29**, 104–132 (1978).

[6] Weyl, H.: Theory of Groups and Quantum Mechanics. Dover (1950).