30. *On Eisenstein Series of Degree Two*

By Nobushige KUROKAWA and Shin-ichiro MIZUMOTO
Department of Mathematics, Tokyo Institute of Technology

(Communicated by Kunihiko KODAIRA, M. J. A., Feb. 12, 1981)

Introduction. We report arithmetical results on Eisenstein series of degree two. There exist two types of Eisenstein series of degree two. The first is the original Eisenstein series studied by Siegel [14] [15] and Maass [9] [10]. The second is the Eisenstein series attached to an elliptic cusp form constructed by Langlands [8] and Klingen [2]. We describe properties of Eisenstein series of degree two concerning the action of Hecke operators and the Fourier coefficients in a unified form. These results were motivated by [4] where certain arithmetical properties of Eisenstein series of degree two (of two types) were examined in connection with congruences between Siegel modular forms.

§ 1. Eisenstein series of degree two. We denote by $M_k(\Gamma_n)$ (resp. $S_k(\Gamma_n)$) the vector space over the complex number field $\boldsymbol{C}$ consisting of all Siegel modular (resp. cusp) forms of degree n and weight k for integers $n \geqq 0$ and $k \geqq 0$. We understand that $M_k(\Gamma_0)=S_k(\Gamma_0)=\boldsymbol{C}$ as usual. The space of Eisenstein series is denoted by $E_k(\Gamma_n)$ which is the orthogonal complement of $S_k(\Gamma_n)$ in $M_k(\Gamma_n)$ with respect to the Petersson inner product. Each modular form f in $M_k(\Gamma_n)$ has the Fourier expansion of the following form: $f=\sum_{T\geqq 0} a(T, f)q^T$ with $q^T=\exp(2\pi\sqrt{-1}\cdot \mathrm{trace}(TZ))$ where Z is a variable on the Siegel upper half space of degree n and T runs over all $n\times n$ symmetric semi-integral positive semi-definite matrices. For a subring R of $\boldsymbol{C}$ we put $M_k(\Gamma_n)_R=\{f\in M_k(\Gamma_n) \mid a(T, f)\in R \text{ for all } T\geqq 0\}$ (an R-module). We denote by $\mathrm{Aut}(\boldsymbol{C})$ the group of all field-automorphisms of $\boldsymbol{C}$. For $n\geqq 0$ and even $k\geqq 0$, each $\sigma\in \mathrm{Aut}(\boldsymbol{C})$ acts on $f=\sum_{T\geqq 0} a(T, f)q^T\in M_k(\Gamma_n)$ by $\sigma(f)=\sum_{T\geqq 0}\sigma(a(T, f))q^T\in M_k(\Gamma_n)$. We say that a modular form f in $M_k(\Gamma_n)$ is eigen if f is a non-zero eigenfunction of all Hecke operators on $M_k(\Gamma_n)$. We say that an eigen modular form f in $M_k(\Gamma_1)$ is normalized if $a(1, f)=1$. In this case we put $\boldsymbol{Q}(f)=\boldsymbol{Q}(\{a(n, f) \mid n\geqq 1\})$ and $\boldsymbol{Z}(f)=\boldsymbol{Q}(f)\cap \bar{\boldsymbol{Z}}$, where $\boldsymbol{Q}$ is the rational number field, $\boldsymbol{Z}$ is the rational integer ring, and $\bar{\boldsymbol{Z}}$ is the ring of all algebraic integers in $\boldsymbol{C}$. Note that $a(n, f)$ is the eigenvalue of the usual Hecke operator $T(n)$ for $f: T(n)f=a(n, f)f$. Then $\boldsymbol{Q}(f)$ is a totally real finite extension of $\boldsymbol{Q}$, and $\boldsymbol{Z}(f)$ is the integer ring of $\boldsymbol{Q}(f)$.

In this paper we restrict our attention to the space $E_k(\Gamma_2)$ of Eisenstein series of degree two. Hereafter in this section, k is an even integer $\geqq 4$. The space $E_k(\Gamma_2)$ is constructed from $M_k(\Gamma_1)$ as follows. For each modular form f in $M_k(\Gamma_1)$ we put $[f]=E^k_{2,0}(*, \Phi f)$ (resp. $E^k_{2,1}(*, f)$) if $\Phi f \neq 0$ (resp. $\Phi f=0$). Here $E^k_{2,r}$ $(r=0,1)$ is the Eisenstein series defined in Klingen [2], and $\Phi: M_k(\Gamma_n)\to M_k(\Gamma_{n-1})$ is the Siegel operator. Then $[\ ]: M_k(\Gamma_1)\to M_k(\Gamma_2)$ is a $\boldsymbol{C}$-linear injection satisfying $\Phi([f])=f$ for all $f \in M_k(\Gamma_1)$, and we have $E_k(\Gamma_2)=[M_k(\Gamma_1)]$.

The following two theorems are proved in [6] in a generalized form.

Theorem 1. *Let f be an eigen modular form in $M_k(\Gamma_1)$ for even $k\geqq 4$. Then:*

(1) *$[f]$ is an eigen modular form in $M_k(\Gamma_2)$.*

(2) *Let F be an eigen modular form in $M_k(\Gamma_2)$ satisfying $\Phi(F)=f$. Then $F=[f]$.*

Theorem 2. *Let f be a modular form in $M_k(\Gamma_1)$ for even $k\geqq 4$. Then:*

(1) *For each $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ we have $\sigma([f])=[\sigma(f)]$.*

(2) *Assume that $f \in M_k(\Gamma_1)_K$ for a subfield K of $\boldsymbol{C}$. Then $[f] \in M_k(\Gamma_2)_K$.*

(3) *Assume that f is a normalized eigen modular form. Then there exists a non-zero constant $\gamma \in \boldsymbol{Z}(f)$ such that $\gamma[f] \in M_k(\Gamma_2)_{\boldsymbol{Z}(f)}$. In particular, $[f] \in M_k(\Gamma_2)_{\boldsymbol{Q}(f)}$.*

Remark 1. Let f be a normalized eigen modular form in $M_k(\Gamma_1)$ such that $\Phi(f)\neq 0$. Then we have $f=G_k=-B_k/2k+\sum_{n\geqq 1}\sigma_{k-1}(n)q^n$, $\boldsymbol{Q}(f)=\boldsymbol{Q}$, $\boldsymbol{Z}(f)=\boldsymbol{Z}$, and $[f]=-(B_k/2k)\varphi_k$, where B_k is the k-th Bernoulli number, $\sigma_{k-1}(n)=\sum_{d\mid n} d^{k-1}$, and φ_k is the original Eisenstein series of degree two constructed by Siegel [14]. In this case, Theorem 2(3) is contained in much more precise results obtained by Siegel [14] [15] and Maass [9] [10].

Remark 2. Let f be a normalized eigen cusp form in $S_k(\Gamma_1)$. Then two examples of Theorem 2(3) are treated in [4]: we have $\gamma=71^2 11$ (resp. 7) for $f=\Delta_{20}$ (resp. Δ_{12}) as the "minimal" γ. Moreover [4] suggests that γ can be taken as a factor of the "numerator" of $L_2^*(2k-2, f)$.

§2. An explicit formula of Fourier coefficients. Let f be a normalized eigen modular form in $M_k(\Gamma_1)$ for even $k\geqq 4$. We present an explicit formula of the Fourier coefficient $a(T, [f])$ in a special case where $T>0$ satisfies the following condition: $-|2T|$ is a fundamental discriminant (i.e., $-|2T|$ is equal to the discriminant of the quadratic field $\boldsymbol{Q}(\sqrt{-|2T|})$) where $|2T|=\det(2T)$. We use the following notations. Let $L_2(s, f)=\zeta(2s-2k+2)\sum_{n\geqq 1} a(n^2, f)n^{-s}$ be the second L-function attached to f, $\vartheta_T=\sum_{n\geqq 0} b(n, T)q^n$ the ϑ-function attached to T where $b(n, T)=\#\left\{(x, y)\in \boldsymbol{Z}^2 \mid (x\ y)T\begin{pmatrix} x \\ y\end{pmatrix}=n\right\}$ ($\#$ denoting the cardinality),

$D(s, f, \vartheta_T)=\sum_{n\geq 1} a(n, f)b(n, T)n^{-s}$, $\chi_{-|2T|}=(-|2T|/*)$ the Dirichlet character associated with $\boldsymbol{Q}(\sqrt{-|2T|})$ (the Kronecker symbol or the generalized Legendre symbol), and $L(s, \chi_{-|2T|})$ the Dirichlet L-function. Here each L-function (Dirichlet series) is considered as a meromorphic function on $\boldsymbol{C}$ by the analytic continuation.

The proof of the following theorem appears in [12].

Theorem 3. *Let the notations be as above. Then we have:*

$$a(T, [f])=(-1)^{k/2}\frac{(k-1)!}{(2k-2)!}(2\pi)^{k-1}|2T|^{k-(3/2)}\frac{L(k-1, \chi_{-|2T|})D(k-1, f, \vartheta_T)}{L_2(2k-2, f)}.$$

Remark 3. For $f=G_k$ as in Remark 1, Theorem 3 coincides with the explicit formula obtained by Maass [9].

Remark 4. The appearance of $L_2(2k-2, f)$ in the "denominator" was suggested by [4], cf. Remark 2 in § 1.

Remark 5. The other Fourier coefficients are also expressed by the eigenvalues of Hecke operators and L-functions.

§ 3. Applications. We have applications in two directions: (I) degree 2 $\Rightarrow$ degree 1, (II) degree 1 $\Rightarrow$ degree 2. Let f be a normalized eigen modular form in $M_k(\Gamma_1)$ for an even integer $k\geq 4$, and we assume that $T>0$ satisfies the condition in § 2. We put $A(T, f)=D(k-1, f, \vartheta_T)\pi^{2k-2}/L_2(2k-2, f)$. We note applications related to $A(T, f)$.

(I) $[f]\Rightarrow f$: By Theorems 2 and 3 we have $\sigma(A(T, f))=A(T, \sigma(f))$ for each $\sigma\in \mathrm{Aut}(\boldsymbol{C})$. In particular $A(T, f)\in \boldsymbol{Q}(f)$. We note that this gives a new proof for the algebraicity of a quotient of special values of L-functions. Moreover we have the value $A(T, f)$ by calculating the value $a(T, [f])$ as in [3] [4] (e.g., Remark 2 of [4]) where the method of Maass [11] is effective. So, for example, if we have the value $L_2(2k-2, f)$ as in [4], which is obtained by the method of Zagier [17], we have the values $D(k-1, f, \vartheta_T)$ for various T.

(II) $f\Rightarrow[f]$: By combining the results of Shimura [13], Sturm [16], and Zagier [17], we have $\sigma(A(T, f))=A(T, \sigma(f))$ for each $\sigma\in \mathrm{Aut}(\boldsymbol{C})$. Hence, by Theorem 3, we have $\sigma(a(T, [f]))=a(T, [\sigma(f)])$. In particular, $a(T, [f])\in \boldsymbol{Q}(f)$. This is a result contained in Theorem 2. There are further possible applications in this direction in connection with the congruences between Siegel modular forms, where the explicit determination of the "minimal" γ appearing in Theorem 2(3) is crucial; see [4] and Remark 2 in § 1.

The above situation is considered as a special case of the following situation. Let G and H be reductive groups over a number field. Let $M(G)$ (resp. $M(H)$) be a suitable space of modular forms on G (resp. H), and assume that there exist two maps

$$L : M(H) \rightleftarrows M(G) : R$$

such that $R\circ L=I$ is the identity map on $M(H)$ (so L is injective). (In some cases, the left map L is the "lifting" and the right map R is the "reduction".) Then we may expect implications in two directions: (I) $M(G) \Rightarrow M(H)$, (II) $M(H) \Rightarrow M(G)$.

We note two examples in the case of Siegel modular forms.

(A) (Eisenstein series) $G=Sp(n)$ and $H=Sp(r)$ for $n \geqq r \geqq 0$. (Here we follow the classical notation. In another notation, $G=CSp(2n)$ and $H=CSp(2r)$.) We put $M(G)=M_k(\Gamma_n)$ and $M(H)=M_k(\Gamma_r)$ for an even integer $k>n+r+1$. We put $R=\Phi^{n-r}$ and $L=[\ ]^{(n-r)}$ in the notation of [6]. Then $R \circ L=I$ on $M(H)$. The case treated in this paper is $G=Sp(2)$ and $H=Sp(1)$ (containing $k=4$). The "uniformity" of the results in this paper seems to be suggestive for the general case (cf. [6] and [6–II]).

(B) $G=Sp(2)$ and $H=Sp(1)$. We put $M(G)=M_k(\Gamma_2)$ and $M(H)=M_{2k-2}(\Gamma_1)$ for an even integer $k \geqq 4$. For the maps we refer to [3] and [4–II].

The Eisenstein series are constructed by Langlands [8] for general reductive groups. In Langlands [7, § 6] we see many examples analogous to (A) with applications of type (I) (the meromorphy of Eisenstein series $\Rightarrow$ the meromorphy of L-functions). Some results seem to suggest the following problem for general "lifting" $L: M(H) \to M(G)$ over reductive groups containing the map given by the Eisenstein series: *to express the Fourier coefficients of $L(f)$ by using (a system of) L-functions attached to $f \in M(H)$.* When this problem is solved affirmatively, we may expect applications of two types (I) and (II) as above. The general conjectures described by Deligne [1] concerning the special values of L-functions would suggest applications of type (II). (For example, we would have interpretations of congruences between modular forms, as noted in the end of [4].)

We note a few numerical examples. We use the notations of [3] [4]. We put $a^*(T)=(7/23)a(T, [\Delta_{12}])/w(T)$, where Δ_{12} denotes the normalized eigen cusp form in $S_{12}(\Gamma_1)$ and $w(T)$ denotes the number of roots of unity in $\boldsymbol{Q}(\sqrt{-|2T|})$. We have the values of $a^*(T)$ in Table I by using the expression of $[\Delta_{12}]$ noted in the proof of Theorem 3 of [4] as in [3].

Table I

n	T_n	$a^*(T_n)$
1	(1, 1, 1)	$2/3$
2	(1, 1, 0)	$3^3/2$
3	(1, 4, 1)	$2^4 3 \cdot 23 \cdot 13171$
4	(2, 2, 1)	$-2^4 3 \cdot 5 \cdot 1327$
5	(1, 5, 0)	$2^2 3^4 5^2 36821$
6	(2, 3, 2)	$-2^2 3^4 8179$

Table II

D	$L^*(D)$
1	$2 \cdot 3^2 5/691$
−3	$2^7 3 \cdot 5 \cdot 7$
−4	$2^5 3^5 5 \cdot 7$
−15	$2^{11} 3^7 5 \cdot 7 \cdot 31 \cdot 59$
−20	$2^9 3^6 5 \cdot 7 \cdot 37 \cdot 12329$
5	$2^{10} 3^4 5 \cdot 7$

We put $L(1)=L(11, \Delta_{12})=\sum_{n\geq 1} a(n, \Delta_{12})n^{-11}$ and $L(D)=L(11, \Delta_{12}, (D/*))=\sum_{n\geq 1}(D/n)a(n, \Delta_{12})n^{-11}$ when $D\in \boldsymbol{Z}$ is equal to the discriminant of the quadratic field $\boldsymbol{Q}(\sqrt{D})$. Then Theorem 3 gives the following correspondences between the Fourier coefficients and the L-functions appearing in the "numerators".

$a(T_1)$	$a(T_2)$	$a(T_3)+a(T_4)$	$a(T_3)-a(T_4)$	$a(T_5)+a(T_6)$	$a(T_5)-a(T_6)$
$L(1)L(-3)$	$L(1)L(-4)$	$L(1)L(-15)$	$L(-3)L(5)$	$L(1)L(-20)$	$L(-4)L(5)$

We put $L^*(D)=(2\pi)^{-11}\cdot 10!\cdot|D|^{11}\cdot L(D)/(\sqrt{|D|}\cdot C_{\mathrm{sign}(-D)})$ where C_+ and C_- are the "periods" of Δ_{12} as in Zagier [17, p. 119] ($C_+=0.0463\cdots$, $C_-=0.0457\cdots$)prescribed by Deligne [1]. Then we have the values of $L^*(D)$ in Table II. We note that the following equality is considered as a self-consistency:

$$a^*(T_1)/a^*(T_2)=(a^*(T_3)-a^*(T_4))/(a^*(T_5)-a^*(T_6)).$$

Both sides are equal to $L^*(-3)/L^*(-4)=2^2/3^4$. Note: $a^*(T_3)-a^*(T_4)=2^{10}3\cdot 7\cdot 691$, $a^*(T_5)-a^*(T_6)=2^8 3^5 7\cdot 691$; $a^*(T_3)+a^*(T_4)=2^5 3^5 31\cdot 59$, $a^*(T_5)+a^*(T_6)=2^3 3^4 37\cdot 12329$.

References

[1] P. Deligne: Valeurs de fonctions L et périodes d'intégrales. Proc. Symp. Pure Math., vol. 33, part 2, pp. 313–346 (1979).

[2] H. Klingen: Zum Darstellungssatz für Siegelsche Modulformen. Math. Z., **102**, 30–43 (1967).

[3] N. Kurokawa: Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two. Invent. Math., **49**, 149–165 (1978).

[4] ——: Congruences between Siegel modular forms of degree two. Proc. Japan Acad., **55**A, 417–422 (1979); II. ibid., **57**A, 140–145 (1981).

[5] ——: On Siegel eigenforms. Proc Japan Acad., **57**A, 47–50 (1981).

[6] ——: On Eisenstein series for Siegel modular groups. Proc. Japan Acad., **57**A, 51–55 (1981); II (preprint).

[7] R. P. Langlands: Euler Products. Yale Univ. Press (1971).

[8] ——: On the functional equations satisfied by Eisenstein series. Lect. Notes in Math., vol. 544, Springer-Verlag (1976).

[9] H. Maass: Die Fourierkoeffizienten der Eisensteinreihen zweiten Grades. Dan. Vid. Selsk., **34**(7) (1964).

[10] ——: Über die Fourierkoeffizienten der Eisensteinreihen zweiten Grades. Dan. Vid. Selsk., **38**(14) (1972).

[11] ——: Lineare Relationen für die Fourierkoeffizienten einiger Modulformen zweiten Grades. Math. Ann., **232**, 163–175 (1978).

[12] S. Mizumoto: Fourier coefficients of generalized Eisenstein series of degree two. Master Thesis, Tokyo Institute of Technology (1981 March).

[13] G. Shimura: The special values of the zeta functions associated with cusp forms. Comm. Pure Appl. Math., **29**, 783–804 (1976).

[14] C. L. Siegel: Einführung in die Theorie der Modulfunktionen n-ten Grades. Math. Ann., **116**, 617–657 (1939).

[15] C. L. Siegel: Über die Fourierschen Koeffizienten der Eisensteinschen Reihen. Dan. Vid. Selsk., **34**(6) (1964).
[16] J. Sturm: Special values of zeta functions and Eisenstein series of half integral weight. Amer. J. Math., **102**, 219–240 (1980).
[17] D. Zagier: Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. Lect. Notes in Math., vol. 627, Springer-Verlag, pp. 105–169 (1977).