3. *The Spectrum of the Laplacian of a Z_3-Invariant Domain*

By Masao TANIKAWA
Institute of Mathematics, University of Tsukuba

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1981)

Introduction. In our previous note [3], the authors proved that for generic bounded domain in R^2, the eigenvalues of the Laplacian Δ with Dirichlet null boundary condition are of multiplicity one. In this paper, we study the eigenvalues of the Laplacian Δ of Z_3-invariant domains $\Omega_\rho \subset R^n$ parametrized by $\rho \in \Gamma$, where the parameter space Γ is an open subset in a Banach (Fréchet) space B.

There are two types of eigenvalues; the symmetric ones whose eigenfunctions u satisfy:

$$u(x)-u(\sigma x)=0,$$

and the anti-symmetric ones whose eigenfunctions u satisfy:

$$u(x)+u(\sigma x)+u(\sigma^2 x)=0,$$

where $\sigma \in SO(n, R)$ is a generator of $Z_3 \subset SO(n, R)$.

A subset of Γ is called residual if it is a countable intersection of open dense subsets. Our main theorem is

Theorem 1. *There exists a residual subset $\Gamma_0 \subset \Gamma$ such that for any $\rho \in \Gamma_0$, all symmetric eigenvalues of Ω_ρ are of multiplicity one and all anti-symmetric eigenvalues of Ω_ρ are of multiplicity two.*

This is a partial answer to the conjecture of V. I. Arnol'd (cf. [1] Hypothesis of Transversality 5.1). Similar results were already obtained by B. H. Driscoll [2] for the operator $(\Delta+\lambda\rho)$ in the unit disc perturbed by some function ρ.

§ 1. Preliminary. Let $Z_3 \subset SO(n, R)$ be a cyclic subgroup of order 3 generated by σ. Let $\Omega \subset R^n$ be a bounded domain with C^r-boundary $\partial\Omega$ $(5 \leq r \leq \infty)$. We assume that Ω is Z_3-invariant. Let $L^2(\Omega) = \left\{u : \Omega \to R, \int_\Omega u(x)^2 dx < \infty\right\}$. $L^2(\Omega)$ is a real Hilbert space with respect to the inner product $(u, v)=\int_\Omega u(x)v(x)dx$. We consider the symmetric subspace $W_s(\Omega)$ and the anti-symmetric subspace $W_a(\Omega)$:

$$W_s(\Omega)=\{u \in L^2(\Omega)\,;\,(1-\sigma^*)u(x)=0 \quad \text{a.e.x.}\},$$

$$W_a(\Omega)=\{u \in L^2(\Omega)\,;\,(1+\sigma^*+(\sigma^2)^*)u(x)=0 \quad \text{a.e.x.}\},$$

where $(\sigma^m)^*u(x)=u(\sigma^m x)$, $m=1, 2$. $L^2(\Omega)$ is orthogonally decomposed into $W_s(\Omega) \oplus W_a(\Omega)$ and the orthogonal projection $\pi : L^2(\Omega) \to W_s(\Omega)$ is equal to $(1+\sigma^*+(\sigma^2)^*)/3$.

Let $B=\{\rho \in C^r(\partial\Omega : R)\,;\,\sigma^*\rho=\rho\}$. We extend $\rho \in B$ to a C^{r-2}-mapping $\tilde{\rho} : R^n \to R^n$ as follows:

$$\tilde{\rho}(x)=\begin{cases}\rho(x)\nu(x) & x\in\partial\Omega,\\ 0 & x\notin N,\end{cases}$$
$$\sigma^{-1}\cdot\tilde{\rho}\cdot\sigma(x)=\tilde{\rho}(x) \qquad x\in R^n,$$

where N is a sufficiently small tublar neighbourhood of $\partial\Omega$ and $\nu(x)$ is the unit outer normal at $x\in\partial\Omega$. Let $\Gamma=\{\rho\in B;\ \|\tilde{\rho}\|_{C^1}<1\}$. Γ is an open set in a Banach (Frécht, if $r=\infty$) space B. For any $\rho\in\Gamma$, $I+\tilde{\rho}$: $R^n\to R^n$ is a C^{r-2}-diffeomorphism and we put $\Omega_\rho=(I+\tilde{\rho})\Omega$. Ω_ρ is a Z_3-invariant domain in R^n and its boundary $\partial\Omega_\rho=\{x+\rho(x)\nu(x);\ x\in\partial\Omega\}$ is of class C^{r-2}.

We define $T=T(\rho): L^2(\Omega_\rho)\to L^2(\Omega)$ by

$$Tu(x)=\sqrt{J(x)}\,u(x+\tilde{\rho}(x)) \qquad x\in\Omega,$$

where $J(x)$ is the Jacobian of $(I+\tilde{\rho})$ at $x\in\Omega$.

Lemma 1. *T is an isomorphism of Hilbert spaces, that is, T is not only bijective but also preserves their inner products. Moreover T is a bijective mapping of $H^k(\Omega_\rho)$ to $H^k(\Omega)$ ($k=1, 2, \cdots, r-3$), where $H^k(\Omega)$ is the Sobolev space of degree k.*

Lemma 2. *σ^* commutes with T, that is, $\sigma^*\cdot T=T\cdot\sigma^*$.*

Lemma 3. *The orthogonal projection $\pi: L^2(\Omega)\to W_s(\Omega)$ commutes with T, namely, $T(W_s(\Omega_\rho))=W_s(\Omega)$ and $T(W_a(\Omega_\rho))=W_a(\Omega)$.*

We introduce a complex structure J on $W_a(\Omega_\rho)$ in the following manner:

$$Ju(x)=1/\sqrt{3}\ (\sigma^*-(\sigma^2)^*)u(x), \qquad u\in W_a(\Omega_\rho).$$

We define a new Hermitian inner product $[u, v]=(u, v)+i(u, Jv)$. Under this complex structure, $\sigma^*u=\exp(2\pi i/3)u$ for any $u\in W_a(\Omega_\rho)$.

Lemma 4. *For any $\rho\in\Gamma$, $W_a(\Omega_\rho)$ is a complex Hilbert space and T preserves its complex structure J.*

§2. Reduction. We consider the eigenvalue problem (P. 1):

$$\text{(P. 1)} \qquad \begin{cases}(-\Delta-\lambda)u(x)=0 & x\in\Omega_\rho,\\ \qquad u|_{\partial\Omega_\rho}=0,\end{cases}$$

where $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\cdots+\partial^2/\partial x_n^2$. Let $\Sigma(\Omega_\rho)=\{\lambda_1\leqq\lambda_2\leqq\lambda_3\leqq\cdots\}$ be the totality of eigenvalues of Problem (P. 1).

Let $L(\rho)$ be the Friedrichs extension of $-T(\rho)\cdot\Delta\cdot T(\rho)^{-1}$ with Dirichlet null boundary condition. $L(\rho)$ is a self-adjoint operator in $L^2(\Omega)$ with domain $\mathcal{D}(L(\rho))=H_0^1(\Omega)\cap H^2(\Omega)$, where $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of C^∞-functions with compact support in Ω. The spectrum of $L(\rho)$ is identical with $\Sigma(\Omega_\rho)$ together with respective multiplicities.

Lemma 5 (cf. [5]). *For any u and $v\in H_0^1(\Omega)\cap H^2(\Omega)$, we have*

$$(L(\rho)u, v)=\int_\Omega\sum_j\sum_k S_{jk}(D_ju)(D_kv)\,dx,$$

where

$$S_{jk}=S_{jk}(\rho)=\sum_m(\delta_{jk}+\partial\chi_j/\partial x_m)(\delta_{km}+\partial\chi_k/\partial x_m),$$
$$I+\chi=(I+\tilde{\rho})^{-1},$$
$$D_ju=\partial u/\partial x_j-(\partial(\log(J(x)))/\partial x_j)u/2.$$

Theorem 2. *$\lambda_j = \lambda_j(\rho)$ is a continuous function of $\rho \in \Gamma$ with respect to C^r-topology* ($5 \leqq r \leqq \infty$).

This is easily proved by using the so-called mini-max principle.

From now on we take account of the group action Z_3. Since the Laplacian Δ commutes with the orthogonal transformation σ and the domain Ω_ρ is Z_3-invariant, we can decompose $\Sigma(\Omega_\rho)$into $\Sigma_s(\Omega_\rho) \cup \Sigma_a(\Omega_\rho)$, where $\Sigma_s(\Omega_\rho)$ and $\Sigma_a(\Omega_\rho)$ are the totalities of eigenvalues of Problem (P. 1) restricted to $W_s(\Omega_\rho)$ and $W_a(\Omega_\rho)$, respectively. We call $\lambda \in \Sigma_s(\Omega_\rho)$ a symmetric eigenvalue and $\lambda \in \Sigma_a(\Omega_\rho)$ an anti-symmetric eigenvalue, respectively. Since $T = T(\rho)$ commutes with the orthogonal projection π, $L(\rho)$ maps $W_s(\Omega)$ and $W_a(\Omega)$ into themselves, respectively. We put $L_s(\rho) = L(\rho)|_{W_s(\Omega)}$ and $L_a(\rho) = L(\rho)|_{W_a(\Omega)}$, respectively.

Lemma 6. *The spectrum of $L_s(\rho)$ (resp. $L_a(\rho)$) is equal to $\Sigma_s(\Omega_\rho)$ (resp. $\Sigma_a(\Omega_\rho)$) together with respective multiplicities.*

By Lemma 6, we are reduced to the study of the spectra of $L_s(\rho)$ and $L_a(\rho)$. Recall that the domains of $L_s(\rho)$ and $L_a(\rho)$ are independent of ρ. For the complex structure J introduced in § 1, we have

Lemma 7. *$L_a(\rho) \cdot J = J \cdot L_a(\rho)$ and $[L_a(\rho)u, v] = [u, L_a(\rho)v]$ for any u and $v \in H^2(\Omega) \cap H_0^1(\Omega) \cap W_a(\Omega)$.*

We consider $L_a(\rho)$ as a complex linear operator $L_c(\rho)$. From Lemma 7, it follows that $L_c(\rho)$ is a self-adjoint operator in $W_a(\Omega)$ with respect to the Hermitian inner product [,].

Lemma 8. *$\Sigma_a(\Omega_\rho)$ is equal to the spectrum* Spec $(L_c(\rho))$ *of $L_c(\rho)$ as a set and the multiplicity of $\lambda \in \Sigma_a(\Omega_\rho)$ is twice of the multiplicity of $\lambda \in$* Spec $(L_c(\rho))$.

§ 3. Proof of the main theorem. Let $S_m = \{\rho \in \Gamma$; the first m spectra of $L_s(\rho)$ are of multiplicity one$\}$ and $T_m = \{\rho \in \Gamma$; the first m spectra of $L_c(\rho)$ are of multiplicity one$\}$. We put $S_0 = T_0 = \Gamma$. Then

$$S_0 \supset S_1 \supset S_2 \supset \cdots; \qquad S = \bigcap_{m=1}^{\infty} S_m,$$

$$T_0 \supset T_1 \supset T_2 \supset \cdots; \qquad T = \bigcap_{m=1}^{\infty} T_m,$$

Theorem 3. *S_m and T_m are open in Γ with respect to C^r-topology* ($5 \leqq r \leqq \infty$), $m = 1, 2, \cdots$.

Theorem 4. *S_m is dense in S_{m-1} with respect to C^r-topology* ($5 \leqq r \leqq \infty$), $m = 1, 2, \cdots$.

Theorem 5. *T_m is dense in T_{m-1} with respect to C^r-topology* ($5 \leqq r \leqq \infty$), $m = 1, 2, \cdots$.

Theorems 3–5 imply that S_m and T_m are open dense in Γ, hence $\Gamma_0 = S \cap T$ is residual. Theorem 3 is an immediate consequence of Theorem 2. The proofs of Theorems 4 and 5 are based on the following perturbation theorem due to Kato [4]: Let $\{H_\tau\}$ be a regular perturbation of self-adjoint operators parametrized by a real parameter τ on some complex Hilbert space. Let H_τ be given formally $H_0+\tau H_1+\tau^2 H_2+\cdots$. Let λ be an isolated spectrum of H_0 with multiplicity q.

Perturbation theorem. i) *For every open interval* $(a, b)\subset R$ *such that* Spec $(H_0)\cap(a, b)=\{\lambda\}$, *there are exactly* q *eigenvalues (counting multiplicity)* $\lambda^1(\tau), \lambda^2(\tau), \cdots, \lambda^q(\tau)$ *of* H_τ *in* (a, b) *where* $\lambda^i(\tau)=\lambda+\tau\lambda_1^i+\tau^2\lambda_2^i+\cdots$ *are real analytic functions for small* $\tau (i=1, 2, \cdots, q)$.

ii) *Let* $\{u^1, u^2, \cdots, u^q\}$ *be an orthonormal basis of* λ*-eigenspace of* H_0. *Then* λ_1^i $(i=1, 2, \cdots, q)$ *are the roots of the equation* det $(\lambda\delta_{jk}-[H_1 u^j, u^k])=0$.

In order to apply Perturbation theorem, we replace $\rho\in\Gamma$ by $\rho_0+\tau\rho$ for sufficiently small $\tau\in R$.

Lemma 9. $L(\rho_0+\tau\rho)$, $L_s(\rho_0+\tau\rho)$ *and* $L_c(\rho_0+\tau\rho)$ *are regular perturbation of* τ *on* $L^2(\Omega)\otimes C$, $W_s(\Omega)\otimes C$ *and* $W_a(\Omega)$, *respectively.*

Lemma 10 (cf. [5]). *Let* u *and* v *be* λ*-eigenfunctions of* $L(0)$. *Then we have*

$$\frac{d}{d\tau}(L(\tau\rho)u, v)|_{\tau=0}=-\int_{\partial\Omega}\rho(x)\frac{\partial u}{\partial\nu}(x)\frac{\partial v}{\partial\nu}(x)\, d\omega(x),$$

where $d\omega(x)$ *is the surface element of* $\partial\Omega$.

Proof of Theorems 4 and 5. We shall show that S_{m+1} (resp. T_{m+1}) is dense in S_m (resp. T_m) for $m=1, 2, \cdots$. Assume that $\rho_0\in S_m$ (resp. T_m) is given. Since $(I+\tilde{\rho}_0+\tilde{\rho})\Omega=(I+\tilde{\theta})\Omega_{\rho_0}$ for some θ, we can replace Ω_{ρ_0} by Ω and $\Omega_{\rho_0+\rho}$ by Ω_θ, respectively. Thus we may assume $\rho_0=0$. Suppose that

$$\lambda_1<\lambda_2<\cdots<\lambda_m=\lambda_{m+1}=\cdots=\lambda_{m+q}<\lambda_{m+q+1}\leqq\cdots$$

are the spectra of $L_s(0)$ (resp. $L_c(0)$). The first m spectra are simple and will remain simple under small perturbations of ρ by Theorem 2. The $(m+1)$-th spectrum $\lambda(=\lambda_{m+1}=\cdots=\lambda_{m+q})$ has multiplicity q. We show that there is a linear perturbation $\rho(\tau)=\tau\rho$ such that the $(m+1)$-th spectrum of $L_s(\tau\rho)$ (resp. $L_c(\tau\rho)$) has multiplicity $\leqq q-1$ for small $\tau\neq 0$. By a finite sequence of perturbations of this type, the $(m+1)$-th spectrum can be made simple. By Perturbation theorem, it is sufficient to show that λ_1^i are not all the same.

For the proof of Theorem 4, we have only to consider the real Hilbert space $W_s(\Omega)$. Let u^j and u^k be λ-eigenvectors of $L_s(0)$. Note that u^j and u^k are σ^*-invariant $(j, k=1, 2, \cdots, q)$.

$$\mu_{jk}=\frac{d}{d\tau}(L_s(\tau\rho)u^j, u^k)|_{\tau=0}$$
$$=-\int_{\partial\Omega}\rho(x)\frac{\partial u^j}{\partial\nu}(x)\frac{\partial u^k}{\partial\nu}(x)\, d\omega(x).$$

If the equation det $(\lambda\delta_{jk}-\mu_{jk})=0$ only has a q-ple root α, then $\mu_{jk}=\alpha\delta_{jk}$. If $\mu_{jk}\neq 0$ $(j\neq k)$, then at least two of the roots are distinct.

We assume that $\mu_{jk}=0$ $(j\neq k)$ for any $\rho\in\Gamma$. Then

$$\frac{\partial u^j}{\partial \nu}(x)\frac{\partial u^k}{\partial \nu}(x)=0 \qquad \text{for any } x \in \partial\Omega,$$

which yields a contradiction to the fact that u^j and u^k are λ-eigenfunction of Problem (P. 1) (cf. [3]).

For the proof of Theorem 5, we have only to consider the complex Hilbert space $W_a(\Omega)$. Let u^j and u^k be λ-eigenvectors of $L_c(0)$ ($j, k=1, 2, \cdots, q$). Note that u^j and u^k satisfy:

$$u(x)+u(\sigma x)+u(\sigma^2 x)=0.$$

$$\begin{aligned}\mu_{jk} &= \frac{d}{d\tau}[L_c(\tau\rho)u^j, u^k]|_{\tau=0}\\ &= \frac{d}{d\tau}(L_a(\tau\rho)u^j, u^k)|_{\tau=0}+i\frac{d}{d\tau}(L_a(\tau\rho)u^j, Ju^k)|_{\tau=0}\\ &= -\frac{1}{3}\int_{\partial\Omega}\rho(x)\Big\{\frac{\partial u^j}{\partial \nu}(x)\frac{\partial u^k}{\partial \nu}(x)+\frac{\partial u^j}{\partial \nu}(\sigma x)\frac{\partial u^k}{\partial \nu}(\sigma x)\\ &\qquad +\frac{\partial u^j}{\partial \nu}(\sigma^2 x)\frac{\partial u^k}{\partial \nu}(\sigma^2 x)\Big\}d\omega(x)\\ &\quad +i/\sqrt{3}\int_{\partial\Omega}\rho(x)\Big\{\frac{\partial u^j}{\partial \nu}(x)\Big(\frac{\partial u^k}{\partial \nu}(\sigma x)-\frac{\partial u^k}{\partial \nu}(\sigma^2 x)\Big)\\ &\qquad +\frac{\partial u^j}{\partial \nu}(\sigma x)\Big(\frac{\partial u^k}{\partial \nu}(\sigma^2 x)-\frac{\partial u^k}{\partial \nu}(x)\Big)\\ &\qquad +\frac{\partial u^j}{\partial \nu}(\sigma^2 x)\Big(\frac{\partial u^k}{\partial \nu}(x)-\frac{\partial u^k}{\partial \nu}(\sigma x)\Big)\Big\}d\omega(x).\end{aligned}$$

If for any $\rho \in \Gamma$, $\mu_{jk}=0$ ($j\neq k$), then following three equations hold for any $x \in \partial\Omega$:

$$\frac{\partial u^j}{\partial \nu}(x)+\frac{\partial u^j}{\partial \nu}(\sigma x)+\frac{\partial u^j}{\partial \nu}(\sigma^2 x)=0,$$

$$\frac{\partial u^j}{\partial \nu}(x)\frac{\partial u^k}{\partial \nu}(x)+\frac{\partial u^j}{\partial \nu}(\sigma x)\frac{\partial u^k}{\partial \nu}(\sigma x)+\frac{\partial u^j}{\partial \nu}(\sigma^2 x)\frac{\partial u^k}{\partial \nu}(\sigma^2 x)=0,$$

$$\begin{aligned}&\frac{\partial u^j}{\partial \nu}(x)\Big\{\frac{\partial u^k}{\partial \nu}(\sigma x)-\frac{\partial u^k}{\partial \nu}(\sigma^2 x)\Big\}+\frac{\partial u^j}{\partial \nu}(\sigma x)\Big\{\frac{\partial u^k}{\partial \nu}(\sigma^2 x)-\frac{\partial u^k}{\partial \nu}(x)\Big\}\\ &\quad +\frac{\partial u^j}{\partial \nu}(\sigma^2 x)\Big\{\frac{\partial u^k}{\partial \nu}(x)-\frac{\partial u^k}{\partial \nu}(\sigma x)\Big\}=0.\end{aligned}$$

These are linear homogeneous equations with respect to $((\partial u^j/\partial \nu)(x)$, $(\partial u^j/\partial \nu)(\sigma x)$, $(\partial u^j/\partial \nu)(\sigma^2 x))$ and have a non-trivial solution for any x in some open set$\subset\partial\Omega$. Then the determinant of the equations is equal to:

$$-2\Big\{\Big(\frac{\partial u^k}{\partial \nu}(x)\Big)^2+\Big(\frac{\partial u^k}{\partial \nu}(\sigma x)\Big)^2+\Big(\frac{\partial u^k}{\partial \nu}(\sigma^2 x)\Big)^2\Big\}=0$$

on some open set in $\partial\Omega$, which yields a contradiction to the fact that u^k is a λ-eigenfunction of Problem (P. 1).

References

[1] Arnol'd, V. I.: Modes and quasi-modes. Funct. Anal. Appl., **6**, 94–101 (1972).

[2] Driscoll, B. H.: The multiplicity of the eigenvalues of a symmetric drum. Thesis, Northwestern Univ., Illinois (1978).

[3] Fujiwara, D., Tanikawa, M., and Yukita, S.: The spectrum of the Laplacian and boundary perturbation. I. Proc. Japan Acad., **54A**, 87–91 (1978).

[4] Kato, T.: On the convergence of the perturbation method. J. Fac. Sci. Univ. Tokyo, **6**, 145–226 (1949).

[5] Micheletti, A. M.: Perturbazione dello Spettro doll'operatore di Laplace, in relazione ad una variazione del campo. Ann. Scuola Norm. Sup. Pisa, **26**, 151–169 (1972).