23. On the Equivalence of Certain Comparison Theorems for Holonomic Systems of Differential Equations

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(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1981)

Introduction. In relation to the regular singularities in the theory of linear partial differential equations, J.-P. Ramis [6] and Z. Mebkhout [4] proposed three equivalent conditions of "GAGA" type for holonomic systems of linear differential equations. The purpose of this note is to show how these conditions are related to each other in the framework of duality between holonomic systems of differential equations and their solution sheaves as is developed in a recent work of M. Kashiwara and T. Kawai [2].

Let X be a complex manifold with structure sheaf $\mathcal{O}_X$. We denote by $\mathcal{D}_X$ (resp. by $\mathcal{D}_X^\infty$) the Ring of linear partial differential operators of finite order (resp. of infinite order) with holomorphic coefficients.

Let M be a bounded complex of (left) $\mathcal{D}_X$-Modules with holonomic cohomology and let Y be a closed analytic subset of X. We will give a simple proof of the equivalence of the following three conditions:

(i) The natural morphism

$$\alpha:\ \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M,\mathcal{O}_X)_{X|Y}\longrightarrow \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M,\mathcal{O}_{X\hat{|}Y})$$

is an isomorphism, where $\cdot_{X|Y}$ is the functor i_*i^{-1}, i being the inclusion mapping $Y\hookrightarrow X$, and $\cdot_{X\hat{|}Y}$ is the formal completion along Y.

(ii) The natural morphism

$$\beta:\ \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X,\mathbf{R}\underline{\Gamma}_{[Y]}(M))\longrightarrow \mathbf{R}\underline{\Gamma}_Y(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X,M))$$

is an isomorphism, where $\mathbf{R}\underline{\Gamma}_{[Y]}(M)$ is the algebraic local cohomology of M with supports in Y. (See Kashiwara [1].)

(iii) The natural morphism

$$\gamma:\ \mathcal{D}_X^\infty\underset{\mathcal{D}_X}{\otimes}\mathbf{R}\underline{\Gamma}_{[Y]}(M)\longrightarrow \mathbf{R}\underline{\Gamma}_Y(\mathcal{D}_X^\infty\underset{\mathcal{D}_X}{\otimes}M)$$

is an isomorphism.

In § 1, we recall some fundamental results concerning the duality for holonomic $\mathcal{D}_X$-Modules from Kashiwara and Kawai [2]. We also review, in § 2, the algebraic local cohomology of $\mathcal{D}_X$-Modules and an adjunction formula due to J.-P. Ramis. In the final section, we establish the theorem of equivalence described above, using the results recalled in the preceding sections.

In this note, "holonomic" will mean "coherent holonomic".

§ 1. Duality for holonomic $\mathcal{D}_X$-Modules. Let M be a bounded complex of $\mathcal{D}_X$-Modules. According to Ramis' notation, we define the solution complex and the De Rham complex of M by the formulae

$$\underline{\mathrm{Sol}}(M)=\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M, \mathcal{O}_X) \quad \text{and} \quad \underline{\mathrm{DR}}(M)=\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X, M),$$

respectively.

If M has holonomic cohomology, both $\underline{\mathrm{Sol}}(M)$ and $\underline{\mathrm{DR}}(M)$ are cohomologically bounded complexes of $\mathbf{C}_X$-Modules with constructible cohomology (Kashiwara's finiteness theorem). Moreover, we have

Theorem 1.1 (Mebkhout [3] and Kashiwara-Kawai [2]). *If M is a bounded complex of $\mathcal{D}_X$-Modules with holonomic cohomology, we have two canonical isomorphisms*

$$\underline{\mathrm{Sol}}(M) \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\underline{\mathrm{DR}}(M), \mathbf{C}_X)$$

and

$$\underline{\mathrm{DR}}(M) \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\underline{\mathrm{Sol}}(M), \mathbf{C}_X).$$

For a $\mathbf{C}_X$-Module F, the $\mathbf{C}_X$-Module $\underline{\mathrm{Hom}}_{\mathbf{C}_X}(F, \mathcal{O}_X)$ has a natural structure of $\mathcal{D}_X^\infty$-Module. Taking the right derived functor, we define

$$\underline{\mathrm{Rec}}(F)=\mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(F, \mathcal{O}_X)$$

for any complex F of $\mathbf{C}_X$-Modules. Using the kernel theorem for constructible $\mathbf{C}_X$-Modules, M. Kashiwara and T. Kawai proved the following duality theorem between holonomic $\mathcal{D}_X$-Modules and their solution sheaves.

Theorem 1.2 ("Reconstruction Theorem", Kashiwara-Kawai [2]). *If M is a bounded complex of $\mathcal{D}_X$-Modules with holonomic cohomology, then we have a canonical isomorphism*

$$\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} M \xrightarrow{\sim} \underline{\mathrm{Rec}} \circ \underline{\mathrm{Sol}}(M).$$

From the two duality Theorems 1.1 and 1.2, we obtain a theorem of invariance of the De Rham cohomology with respect to the Ring extension to $\mathcal{D}_X^\infty$.

Theorem 1.3. *If M is a bounded complex of $\mathcal{D}_X$-Modules with holonomic cohomology, we have a canonical isomorphism*

$$\underline{\mathrm{DR}}(M) \xrightarrow{\sim} \underline{\mathrm{DR}}(\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} M).$$

To prove Theorem 1.3, we need the following

Lemma. *If N is a complex bounded above of $\mathcal{D}_X$-Modules and if F is a complex bounded above of $\mathbf{C}_X$-Modules, then we have a canonical isomorphism*

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(N, \underline{\mathrm{Rec}}(F)) \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\mathrm{F}, \underline{\mathrm{Sol}}(N)).$$

Lemma can be proved in a standard way from a "well-known" adjunction formula.

Proof of Theorem 1.3. By Theorem 1.2 and Lemma, we get the isomorphisms

$$\underline{\mathbf{R}\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} M) \xrightarrow{\sim} \underline{\mathbf{R}\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X, \underline{\mathrm{Rec}} \circ \underline{\mathrm{Sol}}(M)) \xrightarrow{\sim} \underline{\mathbf{R}\mathrm{Hom}}_{\mathbf{C}_X}(\underline{\mathrm{Sol}}(M), \underline{\mathrm{Sol}}(\mathcal{O}_X)).$$

In other words,

$$\underline{\mathrm{DR}}(\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} M) \xrightarrow{\sim} \underline{\mathbf{R}\mathrm{Hom}}_{\mathbf{C}_X}(\underline{\mathrm{Sol}}(M), \mathbf{C}_X).$$

Comparing this isomorphism with the second one in Theorem 1.1, we obtain the desired isomorphism.

§2. **Algebraic local cohomology of $\mathcal{D}_X$-Modules.** Let Y be a closed analytic subset of X. We refer to Kashiwara [1] for the definition of algebraic local cohomology of $\mathcal{D}_X$-Modules with supports in Y.

The crucial result on the algebraic local cohomology is the following holonomicity theorem due to M. Kashiwara.

Theorem 2.1 (Kashiwara [1]). *If M is a bounded complex of $\mathcal{D}_X$-Modules with holonomic cohomology, then the complex $\mathbf{R}\underline{\Gamma}_{[Y]}(M)$ has holonomic cohomology.*

The following theorem due to J.-P. Ramis [6] is a key to the duality arguments in §3.

Theorem 2.2 (Ramis [6]). *If M is a bounded complex of $\mathcal{D}_X$-Modules with coherent cohomology, we have a canonical isomorphism*

$$\underline{\mathbf{R}\mathrm{Hom}}_{\mathcal{D}_X}(\mathbf{R}\underline{\Gamma}_{[Y]}(M), \mathcal{O}_X) \xleftarrow{\sim} \underline{\mathbf{R}\mathrm{Hom}}_{\mathcal{D}_X}(M, \mathcal{O}_{X\hat{|}Y}).$$

We remark that Theorem 2.2 can be generalized to an adjunction formula between the algebraic local cohomology and the formal completion of $\mathcal{D}_X$-Modules. (Noumi [5].)

§3. **Theorem of equivalence.** Let Y be a closed analytic subset of X. Then, for each bounded complex M of $\mathcal{D}_X$-Modules, there is a canonical morphism

$$\alpha: \underline{\mathbf{R}\mathrm{Hom}}_{\mathcal{D}_X}(M, \mathcal{O}_X)_{X|Y} \longrightarrow \underline{\mathbf{R}\mathrm{Hom}}_{\mathcal{D}_X}(M, \mathcal{O}_{X\hat{|}Y}).$$

Via the isomorphism in Theorem 2.2, we can replace this morphism by

$$\bar{\alpha}: \underline{\mathbf{R}\mathrm{Hom}}_{\mathcal{D}_X}(M, \mathcal{O}_X)_{X|Y} \longrightarrow \underline{\mathbf{R}\mathrm{Hom}}_{\mathcal{D}_X}(\mathbf{R}\underline{\Gamma}_{[Y]}(M), \mathcal{O}_X),$$

which we write, with the notation of §1, as

$$\bar{\alpha}: \underline{\mathrm{Sol}}(M)_{X|Y} \longrightarrow \underline{\mathrm{Sol}}(\mathbf{R}\underline{\Gamma}_{[Y]}(M)).$$

The other two comparison morphisms are given by

$$\beta: \underline{\mathrm{DR}}(\mathbf{R}\underline{\Gamma}_{[Y]}(M)) \longrightarrow \mathbf{R}\underline{\Gamma}(\underline{\mathrm{DR}}(M))$$

and

$$\gamma: \mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} \mathbf{R}\underline{\Gamma}_{[Y]}(M) \longrightarrow \mathbf{R}\underline{\Gamma}_Y(\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} M),$$

respectively.

In what follows, we assume that M has holonomic cohomology. Then by Theorem 2.1 $\mathbf{R}\underline{\Gamma}_{[Y]}(M)$ is a cohomologically bounded complex with holonomic cohomology.

First, we have a commutative diagram in the derived category

$$\text{(ii)}\Leftarrow\text{(i)}\qquad \begin{array}{ccc} \underline{\mathrm{DR}}(\mathbf{R}\underline{\Gamma}_{[Y]}(M)) & \xrightarrow{\ \sim\ } & \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\underline{\mathrm{Sol}}(\mathbf{R}\underline{\Gamma}_{[Y]}(M)), \mathbf{C}_X) \\ \beta\big\downarrow & & \big\downarrow\ \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\bar{\alpha}, \mathbf{C}_X) \\ \mathbf{R}\underline{\Gamma}_Y(\underline{\mathrm{DR}}(M)) & \xrightarrow{\ \sim\ } & \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\underline{\mathrm{Sol}}(M)_{X|Y}, \mathbf{C}_X), \end{array}$$

where the horizontal arrows are isomorphisms by Theorem 1.1. Dualizing this diagram by Verdier's biduality for constructible $\mathbf{C}_X$-Modules, we get another diagram

$$\text{(i)}\Leftarrow\text{(ii)}\qquad \begin{array}{ccc} \underline{\mathrm{Sol}}(M)_{X|Y} & \xrightarrow{\ \sim\ } & \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\mathbf{R}\underline{\Gamma}_Y(\underline{\mathrm{DR}}(M)), \mathbf{C}_X) \\ \bar{\alpha}\big\downarrow & & \big\downarrow\ \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\beta, \mathbf{C}_X) \\ \underline{\mathrm{Sol}}(\mathbf{R}\underline{\Gamma}_{[Y]}(M)) & \xrightarrow{\ \sim\ } & \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{C}_X}(\underline{\mathrm{DR}}(\mathbf{R}\underline{\Gamma}_{[Y]}(M)), \mathbf{C}_X). \end{array}$$

On the other hand, we have a diagram

$$\text{(iii)}\Leftarrow\text{(i)}\qquad \begin{array}{ccc} \mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} \mathbf{R}\underline{\Gamma}_{[Y]}(M) & \xrightarrow{\ \sim\ } & \underline{\mathrm{Rec}}(\underline{\mathrm{Sol}}(\mathbf{R}\underline{\Gamma}_{[Y]}(M))) \\ \gamma\big\downarrow & & \big\downarrow\ \underline{\mathrm{Rec}}(\bar{\alpha}) \\ \mathbf{R}\underline{\Gamma}_Y(\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} M) & \xrightarrow{\ \sim\ } & \underline{\mathrm{Rec}}(\underline{\mathrm{Sol}}(M)_{X|Y}), \end{array}$$

by Reconstruction Theorem 1.2.

The last diagram we need is given by Theorem 1.3:

$$\text{(ii)}\Leftarrow\text{(iii)}\qquad \begin{array}{ccc} \underline{\mathrm{DR}}(\mathbf{R}\underline{\Gamma}_{[Y]}(M)) & \xrightarrow{\ \sim\ } & \underline{\mathrm{DR}}(\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} \mathbf{R}\underline{\Gamma}_{[Y]}(M)) \\ \beta\big\downarrow & & \big\downarrow\ \underline{\mathrm{DR}}(\gamma) \\ \mathbf{R}\underline{\Gamma}_Y(\underline{\mathrm{DR}}(M)) & \xrightarrow{\ \sim\ } & \underline{\mathrm{DR}}(\mathbf{R}\underline{\Gamma}_Y(\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} M)). \end{array}$$

Now, the following theorem of equivalence is clear.

Theorem 3.1. *Let Y be a closed analytic subset of X. If M is a bounded complex of $\mathcal{D}_X$-Modules with holonomic cohomology, then the following three conditions are equivalent:*

(i) $\alpha: \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M, \mathcal{O}_X)_{X|Y} \longrightarrow \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(M, \mathcal{O}_{X\hat{|}Y})$ *is an isomorphism.*

(ii) $\beta: \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X, \mathbf{R}\underline{\Gamma}_{[Y]}(M)) \longrightarrow \mathbf{R}\underline{\Gamma}_Y(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X, M))$ *is an isomorphism.*

(iii) $\gamma: \mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} \mathbf{R}\underline{\Gamma}_{[Y]}(M) \longrightarrow \mathbf{R}\underline{\Gamma}_Y(\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} M)$ *is an isomorphism.*

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