24. *Monodromy Preserving Deformation of Linear Differential Equations with Irregular Singular Points*

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§ 1. **Introduction.** The purpose of the present article is to study the monodromy preserving deformation of linear ordinary differential equations with irregular singular points.

The theory of monodromy preserving deformation originates in the classical works of continental mathematicians in the beginning of this century (L. Schlesinger [1], R. Fuchs [2], R. Garnier [3]). In particular they revealed that the Painlevé equations are nothing other than the deformation equations for appropriate linear differential equations. Their works, however, had somehow been forgotten until interest is aroused quite recently both from mathematical side (K. Aomoto [4], K. Okamoto [5], B. Klares [6]) and from physico-mathematical side (J. Myers [7], Wu *et al.* [8], Sato *et al.* [9], Ablowitz *et al.* [10], Flaschka-Newell [11]). Since most of the problems appearing in applied mathematics or mathematical physics have irregular singular points, it seems important, not only from theoretical viewpoints but also for applications, to establish a general theory that will cover the cases admitting irregular as well as regular singularities.

The equations considered in this paper are $n\times n$ first order systems of following types;

$$(1.1)\qquad PY=0,\qquad P=d/dx-\left(\sum_{j=1}^{N}A_j/(x-a_j)+B\right)$$

$$(1.2)\qquad PY=0,\qquad P=xd/dx-(Ax^2+Bx+C)$$

$$(1.3)\qquad PY=0,\qquad P=xd/dx-(x^{-1}E+F+xG).$$

In § 2, after explaining the fundamental result by Y. Shibuya [12] concerning asymptotic solutions of linear ordinary differential equations, we investigate the local monodromy preserving deformation in a general situation. § 3 is devoted to the proof of the main results in § 2. Deformation theories for (1.1)–(1.3) are treated in §§ 4–6, respectively. In these sections, we give a necessary and sufficient condition for (1.1)–(1.3) to be deformed without changing the Stokes multipliers, the global monodromy and appropriate connection matrices. We state that the resulting non-linear systems ("the deformation equations") are completely integrable.

Main results of this and a forthcoming note have been obtained in [14].

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§2. Local monodromy preserving deformation in the general case. Let V be a neighborhood of the infinity of the complex Riemann sphere, and U an open set in C^p. Let $A(x,t)$ be an $n\times n$ matrix, meromorphic in $x\in V$ and holomorphic in $t\in U$ and having a local expansion at $x=\infty$

$$A(x,t)=\sum_{\nu=-\infty}^{m-1} A_\nu(t)x^\nu. \tag{2.1}$$

Here $A_{m-1}(t)$ is assumed to be a diagonal matrix with mutually distinct entries.

In this section, we consider an $n\times n$ first order system with an irregular singular point of rank m at $x=\infty$,

$$PY=0, \qquad P=d/dx-A(x,t). \tag{2.2}$$

This equation has a formal matrix solution as follows:

$$\tilde{Y}(x,t)=\hat{Y}(x,t)x^{D(t)}\exp\left(\sum_{\nu=1}^{m} D_\nu(t)x^\nu\right). \tag{2.3}$$

Here $\hat{Y}(x,t)$ is a formal power series in x^{-1} having the $n\times n$ unit matrix as the leading term, and $D_\nu(t)$ for $1\leqq\nu\leqq m$ and $D(t)$ are diagonal matrices. For example we have

(2.4) $m=1;\quad D_1=A_1,\quad D=A_0^{(+)}$

$m=2;\quad D_2=\frac{1}{2}A_2,\quad D_1=A_1^{(+)},\quad D=A_0^{(+)}+(A_1^{(-)}\{A_1\}_{A_2})^{(+)}$

and so forth. Here, for an $n\times n$ matrix $X=(x_{kl})$, we denote by $X^{(+)}=\mathrm{diag}\,(x_{11},\cdots,x_{nn})$ and $X^{(-)}=X-X^{(+)}$ the diagonal and off-diagonal part of X, respectively. The bracket notation $\{\ \}$ is defined as follows. For a diagonal matrix $\Lambda=\mathrm{diag}\,(\lambda_1,\cdots,\lambda_n)$ (or $\mathrm{diag}\,(\lambda_l)_{1\leqq l\leqq n}$, for short) with mutually distinct entries, we define a matrix $\{X\}_\Lambda$ through

$$\{X\}_{\Lambda,kl}=\begin{cases} 0 & \text{if } k=l \\ \dfrac{x_{kl}}{\lambda_k-\lambda_l} & \text{if } k\neq l. \end{cases} \tag{2.5}$$

We also set $\{X,X'\}_\Lambda=\{[X,X']\}_\Lambda$.

We call $\tilde{Y}(x,t)$ the normalized formal matrix solution for (2.2) at $x=\infty$. Note that it is holomorphic in $t\in U$.

Put $S_{l,\delta}=\left\{x\in V;\ \dfrac{\pi(l-1)}{m}-\delta<\arg x<\dfrac{\pi l}{m}\right\}$ for $1\leqq l\leqq 2m+1$. Here δ is a positive number. It is known (Y. Shibuya [12]) that there exist a sufficiently small δ, a sufficiently small open subset $U'\subset U$ and actual fundamental solution matrices $Y_l(x,t)$, $1\leqq l\leqq 2m+1$, of (2.2) such that Y_l has the asymptotic expansion

(2.6) $$Y_l(x,t)\sim \tilde{Y}(x,t) \quad \text{in } S_{l,\delta},\quad 1\leqq l\leqq 2m+1,$$

valid uniformly in U'. Moreover Y_l is uniquely determined by (2.6) (Balser *et al.* [13]). Note that they are holomorphic in U', and that $Y_{2m+1}=Y_1(xe^{-2\pi i})e^{2\pi iD}$. We call them the normalized solution for (2.2) at $x=\infty$. We define Stokes multipliers C_l, $1\leqq l\leqq 2m$, by

(2.7) $$Y_{l+1}=Y_lC_l.$$

Then the local monodromy of Y_1 at $x=\infty$ is $e^{2\pi iD}C_{2m}^{-1}\cdots C_1^{-1}$. We set "the deformation properties" as follows:

(DP) $$dD=0,\quad dC_l=0,\quad 1\leqq l\leqq 2m.$$

Here d denotes the exterior differentiation with respect to parameters t. Our goal in this section is to find a t-equation satisfied by Y_1 when it is deformed under the condition (DP).

Proposition 1. *If "deformation properties" (DP) hold, $\Omega=dY_1\cdot Y_1^{-1}$ is a meromorphic 1-form in the neighborhood of $x=\infty$ such that*

(2.8) $$\Omega=\sum_{\nu=-\infty}^{m}\sum_{\mu=1}^{l}\Phi_\nu^{(\mu)}(t)x^\nu dt_\mu,$$

(2.9) $$dP=[\Omega,P],\qquad d\Omega=\Omega\wedge\Omega,$$

(2.10) $$\sum_{\mu=1}^{l}\Phi_0^{(\mu)}(t)dt_\mu=\text{the coefficient of } x^0 \text{ in } \hat{Y}\left(\sum_{\nu=1}^{m}dD_\nu x^\nu\right)\hat{Y}^{-1}.$$

Here $dP=-dA(x,t)$, and $[\Omega,P]=\Omega P-P\Omega$. Conversely, if there exists a meromorphic 1-form Ω in the neighborhood of $x=\infty$ satisfying (2.5)–(2.7), *then* (DP) *holds. We remark that* (2.9) *is the integrablity condition of the system $PY=0$, $dY=\Omega Y$.*

§3. The proof of Proposition 1. First we prove (2.8)–(2.10) assuming (DP). Since (DP) guarantees the invariance of the local monodromy of Y_1, we see that $\Omega=dY_1\cdot Y_1^{-1}$ is single-valued. Moreover $dY_l\cdot Y_l^{-1}=dY_{l+1}\cdot Y_{l+1}^{-1}$, $1\leqq l\leqq 2m$ for $dC_l=0$, $1\leqq l\leqq 2m$. Hence Ω has the following asymptotic expansion in all the sectors $S_{l,\delta}$, $1\leqq l\leqq 2m+1$,

$$\Omega\sim d\hat{Y}\cdot\hat{Y}^{-1}+\hat{Y}\left(\sum_{\nu=1}^{m}dD_\nu x^\nu\right)\hat{Y}^{-1}.$$

The right-hand side gives the local expansion of Ω at $x=\infty$, due to the single-valuedness of Ω. Then it is clear that $\Omega=d\hat{Y}\cdot\hat{Y}^{-1}+\hat{Y}\left(\sum_{\nu=1}^{m}dD_\nu x_\nu\right)\hat{Y}^{-1}$ satisfies (2.8)–(2.10).

Conversely we assume that a meromorphic 1-form Ω satisfies (2.8)–(2.10). From (2.9) and (2.10), it follows that $dD=0$, and that the normalized formal matrix solution (2.3) satisfies $P\tilde{Y}=0$, $d\tilde{Y}=\Omega\tilde{Y}$ (K. Ueno [14]). Also, we obtain $P(dY_l-Y_l)=(dP-[\Omega,P])Y_l=0$. Hence there exist matrices B_l $(1\leqq l\leqq 2m+1)$ independent of x such that $dY_l-\Omega Y_l=Y_lB_l$. We show $B_l=0$. Since $\Omega=d\tilde{Y}\cdot\tilde{Y}^{-1}$,

(3.1) $$dY_l\cdot Y_l^{-1}-\Omega\sim 0(x^{-1})\qquad \text{in } S_{l,\delta}.$$

On the other hand,

$$(3.2)\qquad Y_l B_l Y_l^{-1} \sim \hat{Y} x^D \exp\left(\sum_{\nu=1}^{m} D_\nu x^\nu\right) B_l \exp\left(-\sum_{\nu=1}^{m} D_\nu x^\nu\right) x^{-D} Y^{-1} \quad \text{in } S_{l,\delta}.$$

If $\exp\left(\sum_{\nu=1}^{m} D_\nu x^\nu\right) B_l \exp\left(-\sum_{\nu=1}^{m} D_\nu x^\nu\right)$ had a non-vanishing off-diagonal element, it would grow exponentially, for the central angle of $S_{l,\delta}$ is larger than $\frac{\pi}{m}$. Hence B_l must be a diagonal matrix, and

$$(3.3)\qquad Y_l B_l Y_l^{-1} \sim B_l + 0(x^{-1}) \quad \text{in } S_{l,\delta}.$$

Comparing the right-hand side of (3.1) with the one of (3.3), we obtain $B_l=0$. Accordingly $dY_l=\Omega Y_l$, $1\leqq l\leqq 2m+1$. Then it is clear that the Stokes multipliers C_l are invariant. This completes the proof.

§ 4. **Construction of the deformation equations for (1.1).** Let U be an open set in $\boldsymbol{C}^p$. The $n\times n$ coefficient matrices $A_j=A_j(t)$, $1\leqq j\leqq N$, $B(t)$ and the regular singular points $a_j=a_j(t)$, $1\leqq j\leqq N$, are assumed to be holomorphic in U. Assume further that

(I) $B=\operatorname{diag}(b_1(t),\cdots,b_n(t))$ with $b_k(t)\neq b_l(t)$ $(k\neq l)$ for $t\in U$,

(II) the differences of the eigenvalues of $A_j(t)$ are not integers,

(III) $a_i(t)\neq a_j(t)$ $(i\neq j)$ for $t\in U$.

From the theorem of Shibuya (§ 2), we have the normalized matrix solutions Y_l $(1\leqq l\leqq 3)$ at infinity such that $Y_l\sim\hat{Y}$ in S_l. Here $\tilde{Y}=\hat{Y}(x,t)x^{D(t)}\exp(xB(t))$ is the normalized formal matrix solution of (1.1) at $x=\infty$, and S_l $(1\leqq l\leqq 3)$ are appropriate sectors with central angles larger than π. We define the Stokes multipliers C_l $(1\leqq l\leqq 2)$ by $Y_{l+1}=Y_l C_l$.

Near $x=a_j$, Y_1 has a local expression of the form

$$(4.1)\qquad Y_1(x,t)=\Phi_j(x,t)(x-a_j)^{L_j}, \qquad 1\leqq j\leqq N,$$

where Φ_j is holomorphic and invertible near $x=a_j$. We set "the deformation properties" as follows:

$$(\text{DP.I})\qquad dD=0,\quad dC_l=0,\quad 1\leqq l\leqq 2,$$

$$(\text{DP.II})\qquad dL_j=0,\qquad 1\leqq j\leqq N.$$

Theorem 1. *The deformation properties* (DP.I), (DP.II) *hold if and only if* A_j $(1\leqq j\leqq N)$ *and* B *satisfy the following non-linear system*

$$(4.2)\qquad dP=[\Omega,P],\qquad d\Omega=\Omega\wedge\Omega,$$

where Ω *is a rational* 1-*form defined by*

$$(4.3)\qquad \Omega=\sum_{j=1}^{N}\frac{A_j}{x-a_j}da_j+\{dB,A\}_B+xdB,\qquad A=\sum_{j=1}^{N}A_j.$$

(4.2) *is equivalently rewritten into the following completely integrable system*

$$(4.4)\qquad dA_j=\sum_{i(\neq j)}[A_i,A_j]d\log(a_i-a_j)+[\{dB,A\}_B,A_j]+[d(a_jB),A_j],\qquad 1\leqq j\leqq N.$$

We remark that $a_j(t)$ $(1\leqq j\leqq N)$ and $B(t)$ can be regarded as independent variables.

§5. Construction of the deformation equations for (1.2). Let U be an open set in C^p. The $n\times n$ coefficient matrix A, B, C of (1.2) are assumed to be holomorphic in U. Assume further that

(I) $A=\mathrm{diag}\,(a_1(t), \cdots, a_n(t))$ with $a_k(t)\neq a_l(t)$ $(k\neq l)$ for $t\in U$,

(II) the differences of the eigenvalues of $C(t)$ are not integers.

At $x=\infty$, we have the normalized formal matrix solution $\tilde{Y}=\hat{Y}(x,t)x^{D(t)}\exp(x^2A(t)+xB^{(+)}(t))$, and the normalized matrix solution Y_l $(1\leqq l\leqq 5)$ of (1.2) in the sense of §2. Let C_l be the associated Stokes multipliers. Near $x=0$, Y_1 may be expressed in the form, $Y_1(x,t)=\Phi(x,t)x^{L(t)}$, where $\Phi(x,t)$ is holomorphic and invertible matrix near $x=0$. As in §4, we set the deformation properties as follows:

(DP. I) $$dD=0,\ dC_l=0,\ 1\leqq l\leqq 4,$$

(DP.II) $$dL=0.$$

Theorem 2. *The deformation properties* (DP.I), (DP.II) *hold if and only if A, B and C satisfy the following non-linear system*

$$dP=[\Omega, P], \qquad d\Omega=\Omega\wedge\Omega. \tag{5.1}$$

Here

$$\Omega=x^2\Phi+x\Psi+\theta \tag{5.2}$$

$$\Phi=\frac{1}{2}dA, \qquad \Psi=dB^{(+)}+\left\{\frac{1}{2}dA, B\right\}_A \tag{5.3}$$

$$\Theta=\{\Phi, C\}_A+\{\Psi, B\}_A+\mathrm{diag}\left\{\frac{1}{2}\sum_{k\neq l} b_{lk}b_{kl}d\left(\frac{1}{a_l-a_k}\right)\right\}_{1\leqq l\leqq n}$$

(5.1) *is equivalently rewritten into the following completely integrable system*

$$dB=\Psi+[\Theta, B]+[\Psi, C],\ dC=[\Theta, C]. \tag{5.4}$$

We note that A and $B^{(+)}$ can be regarded as independent variables.

§6. Construction of the deformation equations for (1.3). Let U be an open set in C^p. The $n\times n$ coefficient matrices, E, F, and G, of (1.3) are assumed to be holomorphic in U. We assume further that

(I) $G=\mathrm{diag}\,(g_1(t), \cdots, g_n(t))$, $E=K\,\mathrm{diag}\,(e_1(t), \cdots, e_n(t))K^{-1}$ with some K holomorphic in U,

(II) the entries of G and $\tilde{E}=\mathrm{diag}\,(e_k(t))_{1\leqq k\leqq n}$ are mutually distinct, respectively.

At $x=\infty$, we have the normalized formal matrix solution $\tilde{Y}=\hat{Y}(x,t)x^{D^{(\infty)}(t)}\exp(xG(t))$, and the normalized matrix solutions Y_l $(1\leqq l\leqq 3)$ in the sense of §2. To consider the asymptotic behavior of Y_l at $x=0$, we make a transformation $Y=KZ$. Then the system (1.3) is converted into

$$\frac{dZ}{dx}=(x^{-2}\tilde{E}+x^{-1}K^{-1}FK+K^{-1}GK)Z. \tag{6.1}$$

Applying the theorem of Shibuya (§2) to (6.1), we have the normalized formal matrix solution $\tilde{Z}=\hat{Z}(x,t)x^{D^{(0)}(t)}\exp(-x^{-1}\tilde{E}(t))$, and the normalized matrix solutions $Z_l(x,t)$ $(1\leqq l\leqq 3)$ of (6.1) at $x=0$. We define the Stokes multipliers $C_l^{(\infty)}, C_l^{(0)}$ $(1\leqq l\leqq 2)$ by

(6.2) $$Y_{l+1}=Y_l C_l^{(\infty)}, \qquad Z_{l+1}=Z_l C_l^{(0)}.$$

We define further the connection matrix Q by

(6.3) $$Y_1=KZ_1Q.$$

The deformation properties we consider are similar to those in the previous paragraph, except that we now require the connection matrix to be constant as well. Namely we set

(DP.I) $$dD^{(\infty)}=dD^{(0)}=0, \quad dC_l^{(\infty)}=dC_l^{(0)}=0, \quad 1\leqq l\leqq 2,$$

(DP.II) $$dQ=0.$$

Theorem 3. *The deformation properties* (DP.I), (DP.II) *hold if and only if $G, F, \hat{E}$ and K satisfy the following non-linear system*

(6.4) $$dP=[\Omega, P], \qquad d\Omega=\Omega\wedge\Omega$$
$$dK=K\{d\tilde{E}, K^{-1}FK\}_{\tilde{E}}+\{dG, F\}_G K.$$

Here $\Omega=x\Phi+\Psi+x^{-1}\Theta$ is an $n\times n$ matrix 1-form in t given by

(6.5) $$\Phi=dG, \quad \Psi=\{dG, F\}_G, \quad \Theta=-Kd\tilde{E}K^{-1}.$$

The above system (6.4) *is equivalently rewritten into the following completely integrable system*

(6.6) $$dK=K\{d\tilde{E}, K^{-1}FK\}_{\tilde{E}}+dG, F\}_G K,$$
$$dF=[\Phi, E]+[\Theta, G]+[\Psi, F].$$

We note that G and $\tilde{E}$ can be regarded as independent variables.

Further remark. The deformation equation for the type of (1.3) was considered in Sato *et al.* [9], but they did not make clear what is preserved under the deformation.

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