23. *Potential Operators Associated with Absorbing Bessel Processes*

By Junji TAKEUCHI
Department of Mathematics, Ochanomizu University

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1. Introduction. Let α be any real number. Consider a conservative Bessel diffusion process of index α on the half line $\boldsymbol{R}^+=[0,\infty)$ determined by infinitesimal generator

$$A=\frac{1}{2}\left(\frac{d^2}{dx^2}+\frac{\alpha-1}{x}\frac{d}{dx}\right). \tag{1.1}$$

Using the terminology for diffusions described in Itô-McKean [2, p. 130], here is the boundary classification:

(i) For all real α, the end point ∞ is a natural (not exit, not entrance) boundary;

(ii) $\alpha\geqq 2$: 0 is an entrance but not exit point;
$0<\alpha<2$: 0 is an entrance and exit point;
$\alpha\leqq 0$: 0 is an exit but not entrance point.

Particularly in the case $0<\alpha<2$, appropriate boundary condition must be imposed at the origin. The most familiar condition is the case of an instantaneously reflecting barrier. In a previous paper [1], Arakawa and the present author gave the domain and the representation of potential operator for the reflecting Bessel process. The potential kernel obtained there is expressible as $U(x\vee y)$, where

$$U(x)=\begin{cases}\dfrac{1}{\alpha-2}\cdot\dfrac{1}{x^{\alpha-2}} & \text{if } 0<\alpha<2 \text{ and } \alpha>2,\\[2ex] \log\dfrac{1}{x} & \text{if } \alpha=2\end{cases}$$

and $x\vee y$ denotes the greater of x and y.

In this note we deal with the Bessel process of index α $(-\infty<\alpha<2)$ with an absorbing barrier at the origin. More precisely, once a diffusion particle reaches at the origin, it stays there forever. It turns out that the quantity $x\wedge y$, i.e., the smaller of x and y appears in the potential kernels in common. This phenomenon forms a fine contrast to $x\vee y$ in the case of reflecting barrier. Another objective of ours is to study the Bessel process with negative index, for most investigations about Bessel processes have been restricted within the case of positive index.

Associated with the absorbing Bessel process is the infinitesimal generator A given by (1.1) acting on the domain

(1.2) $$\mathcal{D}(A)=C_0(\boldsymbol{R}^+)\cap\{u\,;\,Au\in C_0(\boldsymbol{R}^+),\,u(+0)=0\},$$
where $C_0(\boldsymbol{R}^+)$ is the Banach space of real-valued continuous functions on $\boldsymbol{R}^+$ vanishing at infinity with the norm of uniform convergence. The absorbing condition is analytically expressed as

(1.3) $$u(+0)=0$$

and plays an essential role. The transition probability density $p(t,x,y)$ of the absorbing process is known to be

(1.4) $$p(t,x,y) \\ =C(\alpha)t^{-1}(xy)^{1-\alpha/2}\exp\,[-(x^2+y^2)/2t]y^{\alpha-1}I_{1-\alpha/2}\left(\frac{xy}{t}\right),\qquad t>0,$$

where I_ν is the modified Bessel function. (See Lemma 2 of Molchanov [3].)

In virtue of (1.4), we can check the Sato's criterion [4], i.e.,
$$\lim_{t\to\infty}\int_K p(t,x,y)dy=0$$
for any x and any compact set $K\subset\boldsymbol{R}^+$. Hence the associated semigroup $\{T_t\,;\,t>0\}$ on $C_0(\boldsymbol{R}^+)$ admits the potential operator in Yosida's sense [5]. Namely, the strong limit of the resolvent
$$J_\lambda f=(\lambda-A)^{-1}f=\int_0^\infty e^{-\lambda t}T_tf\,dt\qquad(\lambda>0)$$
as $\lambda\downarrow 0$ exists in a dense subset of $C_0(\boldsymbol{R}^+)$. The linear operator V defined by the limit of J_λ on the domain
$$\mathcal{D}(V)=\left\{f\in C_0(\boldsymbol{R}^+)\,;\,s\text{-}\lim_{\lambda\to 0}J_\lambda f\text{ exists}\right\}$$
is called the potential operator of the semigroup. Then we have $V=-A^{-1}$ and V determines the process uniquely. Further results and earlier literature are referred to in [1].

2. Potential operators for absorbing Bessel processes. We are now ready to prove the theorem on potential operator acting on $C_0(\boldsymbol{R}^+)$.

Theorem. *Let* $-\infty<\alpha<2$. *A function* f *in* $C_0(\boldsymbol{R}^+)$ *belongs to* $\mathcal{D}(V)$ *if and only if*

(2.1) $$\lim_{a\to 0}\left\{\int_a^\infty U(x)x^{\alpha-1}f(x)dx-U(a)\int_a^\infty x^{\alpha-1}f(x)dx\right\}=0.$$

If $f\in\mathcal{D}(V)$, *then*

(2.2) $$Vf(x)=\lim_{a\to 0}\left[-2\left\{\int_a^\infty U(x\wedge y)y^{\alpha-1}f(y)dy-U(a)\int_a^\infty y^{\alpha-1}f(y)dy\right\}\right],$$

here $U(x)$ *has the representation*

(2.3) $$U(x)=\frac{1}{\alpha-2}\cdot\frac{1}{x^{\alpha-2}}$$

and $x\wedge y$ *denotes the smaller of* x *and* y.

Proof. First of all we remark that

(2.4) $$U(x)x^{\alpha-1}=\frac{1}{\alpha-2}x$$

and

$$U'(x)x^{\alpha-1}=-1. \tag{2.5}$$

Set $u=Vf$ for $f\in\mathcal{D}(V)$, then we have $f=V^{-1}u=-Au$ and

$$f(x)=-\frac{1}{2}\left\{u''(x)+\frac{\alpha-1}{x}u'(x)\right\}. \tag{2.6}$$

It follows from (2.6), an integration by parts, (2.5) and (2.4) that

$$\int_a^\infty U(x)x^{\alpha-1}f(x)dx=-\frac{1}{2}\left\{\lim_{x\to\infty}(U(x)x^{\alpha-1}u'(x))-U(a)a^{\alpha-1}u'(a)-u(a)\right\}.$$

On the strength of mean value theorem of differential calculus and boundary condition (1.3), we obtain that $xu'(x)\to 0$ as $x\to\infty$. So we get

$$\lim_{x\to\infty}U(x)x^{\alpha-1}u'(x)=\frac{1}{\alpha-2}\lim_{x\to\infty}xu'(x)=0.$$

Consequently

$$\int_a^\infty U(x)x^{\alpha-1}f(x)dx=\frac{1}{2}\left\{U(a)a^{\alpha-1}u'(a)+u(a)\right\}. \tag{2.7}$$

Applying again (2.6) and an integration by parts, we get

$$\begin{aligned}\int_a^\infty x^{\alpha-1}f(x)dx&=-\frac{1}{2}\left\{\lim_{x\to\infty}(x^{\alpha-1}u'(x))-a^{\alpha-1}u'(a)\right\}\\&=-\frac{1}{2}\left\{\lim_{x\to\infty}\frac{xu'(x)}{x^{2-\alpha}}-a^{\alpha-1}u'(a)\right\}\\&=\frac{1}{2}a^{\alpha-1}u'(a).\end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we see that the expression within the braces in (2.1) is equal to $u(a)$ which tends to zero as $a\to 0$.

In order to prove the converse, suppose f be a function in $C_0(\boldsymbol{R}^+)$ satisfying (2.1). Since we have

$$\begin{aligned}&\int_a^\infty U(x\wedge y)y^{\alpha-1}f(y)dy-U(a)\int_a^\infty y^{\alpha-1}f(y)dy\\&=\left\{U(x)\int_x^\infty y^{\alpha-1}f(y)dy-\int_x^\infty U(y)y^{\alpha-1}f(y)dy\right\}\\&\quad-\left\{U(a)\int_a^\infty y^{\alpha-1}f(y)dy-\int_a^\infty U(y)y^{\alpha-1}f(y)dy\right\},\end{aligned}$$

the right side of (2.2) converges. Define $u(x)$ by its limiting value, then we have

$$\frac{1}{2}u(x)=\int_x^\infty U(y)y^{\alpha-1}f(y)dy-U(x)\int_x^\infty y^{\alpha-1}f(y)dy. \tag{2.9}$$

In virtue of (2.9) and (2.1), it follows that $u\in C_0(\boldsymbol{R}^+)$ and condition (1.3) is satisfied. Further we have the relation (2.6) by differentiating (2.9). These observations imply $u\in\mathcal{D}(A)$, $Au=-f$ and $f\in\mathcal{D}(V)$. According to the general theory of the potential we know $u=Vf$ and that Vf is given by (2.2).

Corollary. *For $-\infty<\alpha<2$, a function f such that $xf(x)\in L^1(\boldsymbol{R}^+)$ is contained in $\mathcal{D}(V)$, and the potential is given by*

$$Vf(x) = -2\int_0^\infty U(x\wedge y)y^{\alpha-1}f(y)dy$$

with kernel (2.3).

Proof. It suffices to check that both the first and the second terms in (2.1) tend to zero. Recalling (2.4) and (2.7), we see that the given assumption is equivalent to $U(x)x^{\alpha-1}f(x) \in L^1(\boldsymbol{R}^+)$ and hence the limit of $U(x)x^{\alpha-1}u'(x)$ as $x\to 0$ exists provided $f\in\mathcal{D}(V)$. One can show the following fact by virtue of mean value theorem of integral calculus: If the integral $\lim_{a\to 0}\int_a^b g(x)dx$ is convergent for a continuous function $g(x)$, there is a sequence $\{x_n\}$ decreasing to zero such that $x_n g(x_n)$ converges to zero. Applying this fact to $\int_0^b u'(x)dx$, we obtain

$$\lim_{a\to 0} U(a)a^{\alpha-1}u'(a) = \frac{1}{\alpha-2}\lim_{a\to 0} au'(a) = 0. \tag{2.10}$$

By means of (2.8) and (2.10) we can conclude

$$\lim_{a\to 0} U(a)\int_a^\infty x^{\alpha-1}f(x)dx = -\frac{1}{2}\lim_{a\to 0} U(a)\cdot a^{\alpha-1}u'(a) = 0.$$

Remark. It is well known that Brownian motion with absorbing boundary condition has the following transition probability density

$$\frac{1}{\sqrt{2\pi t}}\left[\exp\left\{-\frac{(y-x)^2}{2t}\right\} - \exp\left\{-\frac{(y+x)^2}{2t}\right\}\right].$$

So it is foreseen that the quantity $|y-x|-|y+x| = -2(x\wedge y)$ appears in the potential kernel. This relation motivated our research. We give a probabilistic interpretation as Remark 1 in [1]. The terms $|y-x|$ and $|y+x|$ correspond to original Brownian motion and reflection at the origin, respectively. In the present case reflection never occurs, that is, $-|y+x|$ means absorption at the origin.

References

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