22. *Some Remarks on Nonlinear Ergodic Theorems in Banach Spaces*

By Kazuo KOBAYASI*) and Isao MIYADERA**)

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1980)

1. Introduction. Throughout this note we assume that X is a *uniformly convex* real Banach space and C is a *closed convex* nonempty subset of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The *duality mapping* F (multi-valued) from X into X^* will be defined by $F(x)=\{x^* \in X^* : (x, x^*)=\|x\|^2=\|x^*\|^2\}$ for $x \in X$. We say that X is (F) if the norm of X is Fréchet differentiable, i.e. for each $x \in X$ with $x \neq 0$, $\lim_{t\to 0} t^{-1}(\|x+ty\|-\|x\|)$ exists uniformly in $y \in B(0,1)$, where $B(x,r)=\{z \in X : \|z-x\| \leqq r\}$. By $T \in \mathrm{Cont}(C)$ we mean that $T: C \to C$ and T is a *contraction*, i.e. $\|Tx-Ty\| \leqq \|x-y\|$ for x, y in C. The set of fixed points of T will be denoted by $\mathcal{F}(T)$.

Recently R. E. Bruck and S. Reich established a nonlinear mean ergodic theorem, and R. E. Bruck [3] gave a simple proof of the theorem: *If X is (F), C is bounded and $T \in \mathrm{Cont}(C)$, then for each x in C $\{T^n x\}$ is weakly almost-convergent to a point of $\mathcal{F}(T)$.* In this note we deal with the weak almost-convergence of almost-orbits of $\{T_n\}$, $T_n \in \mathrm{Cont}(C)$, and obtain an extension of the above-mentioned theorem (see § 3). To this end we prove Proposition 2.2 in § 2. This proposition is also used to show the weak convergence of almost-orbits of resolvents for m-dissipative operators in § 4.

2. Almost-orbits of contractions. Let $T_n \in \mathrm{Cont}(C)$ for $n \geqq 1$ and set $P_n^m = T_m T_{m-1} \cdots T_n$ for $m \geqq n \geqq 1$. A sequence $\{x_n\}_{n\geqq 0}$ in C is called an *almost-orbit* of $\{T_n\}$ if

$$\lim_{n\to\infty} [\sup_{m\geqq 0} \|x_{n+m} - P_n^{n+m} x_{n-1}\|] = 0.$$

Let Γ be the set of strictly increasing, continuous and convex functions $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0)=0$. According to Bruck [3] we say that $T: C \to X$ is *of type* (γ) if $\gamma \in \Gamma$ and for all $x, y \in C$ and $0 \leqq \lambda \leqq 1$

$$\gamma(\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\|) \leqq \|x-y\| - \|Tx-Ty\|.$$

It is known that if C is bounded then there exists $\gamma \in \Gamma$ such that every contraction $T : C \to X$ is of type (γ). (See [3, Lemma 1.1].)

Lemma 2.1. *Let $T_n \in \mathrm{Cont}(C)$ for $n \geqq 1$, and let $\{x_n\}_{n\geqq 0}$ and $\{y_n\}_{n\geqq 0}$ be almost-orbits of $\{T_n\}$. Then we have the following:*

(a) *$\{\|x_n - y_n\|\}$ is convergent.*

(b) *If $\{y_n\}$ is bounded, then for any $\lambda \in (0,1)$ $\{\lambda x_n + (1-\lambda) y_n\}$ is an*

*) Department of Mathematics, Sagami Institute of Technology, Fujisawa.
**) Department of Mathematics, Waseda University, Tokyo.

almost-orbit of $\{T_n\}$.

Proof. (a) can be proved by the same way as in [3, Lemma 2.1], and therefore we omit the proof of it. Let us now show (b). Put $z_n=\lambda x_n+(1-\lambda)y_n$. Since $\{y_n\}$ is bounded, so is $\{x_n\}$ by (a). Choose an $r>0$ such that $\{x_n\}$, $\{y_n\}\subset B(0,r)$, and put $C_r=C\cap B(0,r)$ and $Q_n^{n+m}=P_n^{n+m}|_{C_r}$ (the restriction of P_n^{n+m} to C_r). Then C_r is a bounded closed convex subset and $Q_n^{n+m}: C_r\to X$ is a contraction. Thus there is $\gamma\in\Gamma$ such that every Q_n^{n+m} is of type (γ). Hence, by a similar way to that of [3, Lemma 2.2] we obtain

$$\|z_{n+m}-Q_n^{n+m}z_{n-1}\| \leqq \lambda\alpha_n+(1-\lambda)\beta_n+\gamma^{-1}(\|x_{n-1}-y_{n-1}\|-\|x_{n+m}-y_{n+m}\|+\alpha_n+\beta_n)$$

for $n\geqq 1$ and $m\geqq 0$, where $\alpha_n=\sup_{m\geqq 0}\|x_{n+m}-Q_n^{n+m}x_{n-1}\|$ and $\beta_n=\sup_{m\geqq 0}\|y_{n+m}-Q_n^{n+m}y_{n-1}\|$. Since $\lim_{n\to\infty}\|x_n-y_n\|$ exists by (a) and $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$, it readily follows that

$$\lim_{n\to\infty}[\sup_{m\geqq 0}\|z_{n+m}-P_n^{n+m}z_{n-1}\|]=0. \qquad \text{Q.E.D.}$$

Proposition 2.2. *Suppose that X is (F),* $T_n\in \mathrm{Cont}(C)$ *for* $n\geqq 1$ *and* $D=\{f\in C: \sum_{n=1}^{\infty}\|T_nf-f\|<\infty\}\neq\emptyset$. *If* $\{x_n\}_{n\geqq 0}$ *is an almost-orbit of* $\{T_n\}$, *then the sequence* $\{(x_n, F(f-g))\}$ *is convergent for all* $f, g\in D$.

Proof. If $f\in D$, the constant sequence $\{f\}$ is an almost-orbit of $\{T_n\}$. Indeed, $\sup_{m\geqq 0}\|f-P_n^{n+m}f\|\leqq\sup_{m\geqq 0}\{\|f-T_{n+m}f\|+\sum_{k=1}^{m}\|P_{n+k}^{n+m}f-P_{n+k-1}^{n+m}f\|\}\leqq\sum_{k=n}^{\infty}\|f-T_kf\|\to 0$ as $n\to\infty$. Let $0<\lambda\leqq 1$ and $f, g\in D$. By Lemma 2.1, $\{\|\lambda x_n+(1-\lambda)f-g\|\}$ is convergent as $n\to\infty$. However, since $\{x_n-f\}$ is bounded, the Fréchet differentiability of the norm of X implies that

$$\lim_{\lambda\downarrow 0}(2\lambda)^{-1}(\|f-g+\lambda(x_n-f)\|^2-\|f-g\|^2)=(x_n-f, F(f-g))$$

uniformly in n. Hence $\lim_{n\to\infty}(x_n-f, F(f-g))=\lim_{n\to\infty,\lambda\downarrow 0}(2\lambda)^{-1}(\|\lambda x_n+(1-\lambda)f-g\|^2+\|f-g\|^2)$ exists. Q.E.D.

Corollary 2.3. *Suppose that X is (F),* $T_n\in\mathrm{Cont}(C)$ *for* $n\geqq 1$ *and* $D=\{f\in C: \sum_{n=1}^{\infty}\|T_nf-f\|<\infty\}\neq\emptyset$. *Let* $\{x_n\}_{n\geqq 0}$ *be an almost-orbit of* $\{T_n\}$, *and let* $\omega_w(\{x_n\})$ *denote the set of weak subsequential limits of* $\{x_n\}$. *Then* $D\cap clco\,\omega_w(\{x_n\})$ *is at most a singleton, where clco E denotes the closed convex hull of E.*

Proof. It follows from Proposition 2.2 that for all $u, v\in\omega_w(\{x_n\})$ and $f, g\in D$, $(u, F(f-g))=\lim_{n\to\infty}(x_n, F(f-g))=(v, F(f-g))$ and hence $(u-v, F(f-g))=0$. But this is also true for all $u, v\in clco\,\omega_w(\{x_n\})$. Thus $D\cap clco\,\omega_w(\{x_n\})$ is at most a singleton. Q.E.D.

3. Weak almost-convergence of almost-orbits. A sequence $\{x_n\}_{n\geqq 0}$ in X is said to be *weakly almost-convergent* to x if $w\text{-}\lim_{n\to\infty}n^{-1}\sum_{k=0}^{n-1}x_{k+i}=x$ uniformly in $i\geqq 0$. By virtue of [3, Theorem 1.1] we have the following

Theorem 3.1. *Suppose that* $T\in\mathrm{Cont}(C)$, $\mathcal{F}(T)\neq\emptyset$ *and* $\{x_n\}_{n\geqq 0}$ *is a bounded sequence in C such that* $\lim_{n\to\infty}\|x_{n+1}-Tx_n\|=0$. *Then for* *every weak neighborhood W of $\mathcal{F}(T)$ there exists a positive integer N such that $n^{-1}\sum_{k=0}^{n-1} x_{k+i} \in W$ for all $n \geq N$ and $i \geq 0$.*

Theorem 3.2. *Let X be (F) and $T, T_n \in \mathrm{Cont}(C)$ for $n \geq 1$. Suppose that*

(i) $\lim_{n\to\infty} T_n x = Tx$ uniformly in $x \in B$ for every bounded set $B \subset C$,

(ii) $\mathcal{F}(T) \neq \emptyset$ and $\mathcal{F}(T) \subset D = \{f \in C : \sum_{n=1}^{\infty} \|T_n f - f\| < \infty\}$.

Then every almost-orbit $\{x_n\}_{n\geq 0}$ of $\{T_n\}$ is weakly almost-convergent to the unique point of $\mathcal{F}(T) \cap clco\, \omega_w(\{x_n\})$.

Proof. Let $i(n)$ be any sequence of nonnegative integers, and set $s_n = n^{-1}\sum_{k=0}^{n-1} x_{k+i(n)}$. It suffices to show that $\{s_n\}$ converges weakly to a point of $\mathcal{F}(T) \cap clco\, \omega_w(\{x_n\})$. Since $\{\|x_n - f\|\}$ is convergent for $f \in D$ (see the proof of Proposition 2.2), $\{x_n\}$ is bounded. Hence, (i) implies that $\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = 0$. Consequently, $\omega_w(\{s_n\}) \subset \mathcal{F}(T)$ by Theorem 3.1. Moreover $\omega_w(\{s_n\}) \subset \bigcap_{i=0}^{\infty} clco\,\{x_k : k \geq i\} = clco\, \omega_w(\{x_n\})$. Thus we have $\omega_w(\{s_n\}) \subset \mathcal{F}(T) \cap clco\, \omega_w(\{x_n\}) \subset D \cap clco\,(\{x_n\})$. However, since $D \cap clco\,(\{x_n\})$ is a singleton by Corollary 2.3, we obtain that $\omega_w(\{s_n\})$ is a singleton and is equal to $\mathcal{F}(T) \cap clco\, \omega_w(\{x_n\})$. Q.E.D.

Remarks. 1) Under the assumptions of Theorem 3.2 we have $\mathcal{F}(T) = D$. 2) If X is a Hilbert space, $x_0 \in C$ and $x_n = T_n x_{n-1}$ for $n \geq 1$ in Theorem 3.2, then the condition (i) can be replaced by a weaker condition "$\lim_{n\to\infty} T_n x = Tx$ for each $x \in C$".

Applying Theorem 3.2 with $T_n = \lambda_n T + (1-\lambda_n)I$, we have the following corollary which extends a result due to S. Reich [4, Note added in Proof]:

Corollary 3.3. *Let X be (F), $T \in \mathrm{Cont}(C)$ and $0 \leq \lambda_n \leq 1$ for $n \geq 1$. If $\mathcal{F}(T) \neq \emptyset$ and $\lim_{n\to\infty} \lambda_n = 1$, then every almost-orbit $\{x_n\}_{n\geq 0}$ of $\{\lambda_n T + (1-\lambda_n)I\}$ is weakly almost-convergent to the unique point of $\mathcal{F}(T) \cap clco\, \omega_w(\{x_n\})$.*

4. Weak convergence of almost-orbits of resolvents. Throughout this section, it is assumed that A is an m-dissipative operator in X with $A^{-1}0 \neq \emptyset$ and $\{\lambda_n\}_{n\geq 1}$ is a positive sequence. For $\lambda > 0$ J_λ denotes the resolvent of A, i.e. $J_\lambda = (I - \lambda A)^{-1}$. Clearly, $J_\lambda : X \to D(A)$ is a contraction.

Lemma 4.1. *If $\{x_n\}_{n\geq 0}$ is an almost-orbit of $\{J_{\lambda_n}\}$ and $y_n = \lambda_n^{-1}(J_{\lambda_n} - I)x_{n-1}$, then $\lim_{n\to\infty} \lambda_n \|y_n\| = 0$.*

Proof. If $f \in A^{-1}0$, the constant sequence $\{f\}$ is an almost-orbit of $\{J_{\lambda_n}\}$ and hence $r = \lim_{n\to\infty} \|x_n - f\|$ exists by Lemma 2.1. If $r = 0$, then $\lambda_n \|y_n\| \leq \|J_{\lambda_n} x_{n-1} - x_n\| + \|x_n - x_{n-1}\| \to 0$. (Note that $\|J_{\lambda_n} x_{n-1} - x_n\| \to 0$, for $\{x_n\}$ is an almost-orbit of $\{J_{\lambda_n}\}$.) Next, let $r > 0$ and choose an n_0 such that $\|x_n - f\| > r/2$ for $n \geq n_0$. Let δ be the modulus of uniform convexity of X. Then $\delta(\|x-y\|) \leq 1 - \|x+y\|/2$ for $x, y \in B(0,1)$. Putting $x = (J_{\lambda_n} x_{n-1} - f)/a_n$ and $y = (x_{n-1} - f)/a_n$, where $a_n = \|x_{n-1} - f\|$, we have

$$a_n\delta(\lambda_n\|y_n\|/a_n)\leqq a_n-\|J_{\lambda_n}x_{n-1}-f-2^{-1}\lambda_n y_n\| \qquad \text{for } n>n_0.$$

However, since $y_n\in AJ_{\lambda_n}x_{n-1}$ and $0\in Af$, the dissipativity of A implies that $\|J_{\lambda_n}x_{n-1}-f\|\leqq\|J_{\lambda_n}x_{n-1}-f-\lambda y_n\|$ for $\lambda>0$. Therefore we have

$$(r/2)\delta(\lambda_n\|y_n\|/M)\leqq\|x_{n-1}-f\|-\|J_{\lambda_n}x_{n-1}-f\| \qquad \text{for } n>n_0,$$

where $M=\sup_{n\geqq1}a_n$. Letting $n\to\infty$, we see that $\lim_{n\to\infty}\lambda_n\|y_n\|=0$.

Q.E.D.

Theorem 4.2. *Suppose that X is (F) and let $\{x_n\}_{n\geqq0}$ be an almost-orbit of $\{J_{\lambda_n}\}$. If $\liminf_{n\to\infty}\lambda_n>0$, then $\{x_n\}$ is weakly convergent to a point of $A^{-1}0$.*

Proof. Put $y_n=\lambda_n^{-1}(J_{\lambda_n}-I)x_{n-1}$. Then $\lambda_n\|y_n\|\to0$ as $n\to\infty$ by Lemma 4.1. Combining this with $\liminf_{n\to\infty}\lambda_n>0$, we obtain $\lim_{n\to\infty}\|y_n\|=0$. Since $y_n\in AJ_{\lambda_n}x_{n-1}$, we have

$$\|(J_1-I)J_{\lambda_n}x_{n-1}\|\leqq\|y_n\|\to0 \qquad \text{as } n\to\infty.$$

Let $\{n'\}$ be a subsequence of $\{n\}$ and $u=w\text{-}\lim_{n'\to\infty}x_{n'}$. Then $u=w\text{-}\lim_{n'\to\infty}J_{\lambda_{n'}}x_{n'-1}$ by $\|x_n-J_{\lambda_n}x_{n-1}\|\to0$ as $n\to\infty$. Therefore by the demiclosedness of J_1-I we have $(J_1-I)u=0$, i.e. $u\in A^{-1}0$. This shows that $\omega_w(\{x_n\})\subset A^{-1}0$. However, since $A^{-1}0\subset D=\{f\in X:\sum_{n=1}^{\infty}\|J_{\lambda_n}f-f\|<\infty\}$, it follows from Corollary 2.3 that $\omega_w(\{x_n\})$ is a singleton. Q.E.D.

Corollary 4.3. *Let $B:X\to X$ be a continuous dissipative operator which maps bounded sets of $D(A)$ into bounded sets in X. Let $\{\varepsilon_n\}_{n\geqq1}$ be a nonnegative sequence and let $x_0\in X$ and $x_n=(I-\lambda_n(A+\varepsilon_nB))^{-1}x_{n-1}$ for $n\geqq1$. (Note that each $A+\varepsilon_nB$ is m-dissipative (see [1]).) Suppose that X is (F) and $\sum_{n=1}^{\infty}\lambda_n\varepsilon_n<\infty$. Then we have the following:*

(a) *If $\liminf_{n\to\infty}\lambda_n>0$, then $\{x_n\}$ is weakly convergent to a point of $A^{-1}0$.*

(b) *If $\limsup_{n\to\infty}\lambda_n>0$ and $\sum_{n=1}^{\infty}|\varepsilon_n-\varepsilon_{n+1}|<\infty$, then $\{x_n\}$ is weakly convergent to a point of $A^{-1}0$.*

Proof. Put $A_n=A+\varepsilon_nB$ and $T_n=(I-\lambda_nA_n)^{-1}$, and let $f\in A^{-1}0$. Then

$$\|x_n-f\|\leqq\|T_nx_{n-1}-T_nf\|+\|T_nf-f\|\leqq\|x_{n-1}-f\|+\lambda_n|||A_nf|||$$
$$\leqq\|x_{n-1}-f\|+\lambda_n\varepsilon_n\|Bf\|, \text{ where } |||A_nf|||=\inf_{z\in A_nf}\|z\|.$$

From this and $\sum_{n=1}^{\infty}\lambda_n\varepsilon_n<\infty$ we see that $\{x_n\}_{n\geqq1}$ is bounded in $D(A)$. Put $K=\sup_{n\geqq1}\|Bx_n\|$. Since $\|x_n-J_{\lambda_n}x_{n-1}\|=\|J_{\lambda_n}(x_{n-1}-\lambda_n\varepsilon_nBx_n)-J_{\lambda_n}x_{n-1}\|\leqq\lambda_n\varepsilon_n\|Bx_n\|\leqq K\lambda_n\varepsilon_n$, we obtain

$$\|x_{n+m}-J_{\lambda_{n+m}}\cdots J_{\lambda_n}x_{n-1}\|\leqq\sum_{k=n}^{n+m}\|x_k-J_{\lambda_k}x_{k-1}\|\leqq K\sum_{k=n}^{\infty}\lambda_k\varepsilon_k$$

for $n\geqq1$ and $m\geqq0$. This shows that $\{x_n\}$ is an almost-orbit of $\{J_{\lambda_n}\}$. Thus (a) is a direct consequence of Theorem 4.2.

To prove (b) it suffices to show that $\lim_{n\to\infty}\|y_n\|=0$ as seen from the proof of Theorem 4.2, where $y_n=\lambda_n^{-1}(J_{\lambda_n}-I)x_{n-1}$. To this end set $v_n=\lambda_n^{-1}(x_n-x_{n-1})$. Since $v_n\in A_nx_n$ and $v_{n+1}+(\varepsilon_n-\varepsilon_{n+1})Bx_{n+1}\in A_nx_{n+1}$, the dissipativity of A_n implies

$$\|v_{n+1}\|\leqq\|v_n\|+|\varepsilon_n-\varepsilon_{n+1}|\,\|Bx_{n+1}\|\leqq\|v_n\|+K|\varepsilon_n-\varepsilon_{n+1}|.$$

Combining this with $\sum_{n=1}^{\infty}|\varepsilon_n-\varepsilon_{n+1}|<\infty$, we have that $\{\|v_n\|\}$ is convergent. Moreover $\|v_n-y_n\|=\lambda_n^{-1}\|x_n-J_{\lambda_n}x_{n-1}\|\leqq K\varepsilon_n\to 0$ as $n\to\infty$ (note that $\{\varepsilon_n\}$ converges by $\sum_{n=1}^{\infty}|\varepsilon_n-\varepsilon_{n+1}|<\infty$). Therefore $\{\|y_n\|\}$ is also convergent, and hence $\lim_{n\to\infty}\|y_n\|=(\limsup_{n\to\infty}\lambda_n)^{-1}(\lim_{n\to\infty}\lambda_n\|y_n\|)=0$. Q.E.D.

Taking $\varepsilon_n=0$ for $n\geqq 1$ in Corollary 4.3 we have the following which is due to S. Reich [4].

Corollary 4.4. *Suppose that X is (F). Let $x_0\in X$ and $x_n=J_{\lambda_n}x_{n-1}$ for $n\geqq 1$. If* $\limsup_{n\to\infty}\lambda_n>0$, *then $\{x_n\}$ is weakly convergent to a point of $A^{-1}0$.*

Remark. In Theorem 4.2 and Corollaries 4.3 and 4.4 the assumption "X is (F)" may be replaced by "X satisfies Opial's condition".

Acknowledgement. We are grateful to Profs. R. E. Bruck and S. Reich who provided us with reprints and preprints of their work.

References

[1] V. Barbu: Continuous perturbations on nonlinear m-accretive operators in Banach spaces. Boll. U.M.I., **6**, 270–278 (1972).
[2] R. E. Bruck: On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω-limit set. Israel J. Math., **29**, 1–16 (1978).
[3] ——: A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces. Ibid., **32**, 107–116 (1979).
[4] S. Reich: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl., **67**, 274–276 (1979).