21. *A Solution to a Problem on the Asymptotic Behavior of Nonexpansive Mappings and Semigroups**)

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Let C be a closed convex subset of a Banach space $E, T: C \to C$ a nonexpansive ($|Tx-Ty| \leqslant |x-y|$ for all x and y in C) mapping, and $S: [0, \infty) \times C \to C$ a nonexpansive nonlinear semigroup. Assume that the norm of E is uniformly Gâteaux differentiable (UG), and that the norm of its dual E^* is Fréchet differentiable (F). It was shown in [5] and [7] that if C is a (sunny) nonexpansive retract of E, then the strong $\lim_{n\to\infty} T^n x/n$ and $\lim_{t\to\infty} S(t)x/t$ exist for each x in C. However, the question whether this is true for arbitrary closed convex subsets of E has remained open [6, Problem 7] and [8, Problem 4]. The purpose of this note is to present a positive solution to this problem (Theorems 2 and 3). Theorem 1 provides a (partial) positive answer to a question of Pazy [4, p. 239].

Recall that a subset A of $E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $|x_1-x_2| \leqslant |x_1-x_2+r(y_1-y_2)|$ for all $[x_i, y_i] \in A$, $i=1, 2$, and $r>0$. The resolvent $J_r: R(I+rA) \to D(A)$ and the Yosida approximation $A_r: R(I+rA) \to R(A)$ are defined by $J_r=(I+rA)^{-1}$ and $A_r=(I-J_r)/r$ respectively. We denote the closure of a subset D of E by $cl(D)$ and its closed convex hull by $clco(D)$. The distance between a point x in E and D is denoted by $d(x, D)$. We shall say that D has the minimum property [4] if $d(0, clco(D))=d(0, D)$. Let J denote the duality map from E to E^*.

Theorem 1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm, and let $A \subset E \times E$ be an accretive operator. If $R(I+rA) \supset cl(D(A))$ for all $r>0$, then $cl(R(A))$ has the minimum property.*

Proof. Let $x \in cl(D(A))$, $z \in Ay$, and $t>0$. Since A is accretive, $(z-A_t x, J((y-J_t x)/t)) \geqslant 0$ for all t. Let a subset of $j_t=J((y-J_t x/t)$ converge weak-star to j as $t \to \infty$. Since we always have $\lim_{t\to\infty} |J_t x/t| = d(0, R(A))=d$ (see the proof of [9, Proposition 5.2]), we see that $|j| \leqslant \liminf_{t\to\infty} |j_t|=d$. We also have $\lim_{t\to\infty} (A_t x, J((y-J_t x)/t))=d^2$. Therefore $(z, j) \geqslant d^2$. Since E is (UG), j does not depend on y and z. Thus

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$(w, j) \geqslant d^2$ for all w in $clco(R(A))$. Hence $|w|\,d \geqslant |w|\,|j| \geqslant (w, j) \geqslant d^2$, and the result follows.

There are examples that show that Theorem 1 is not true for all Banach spaces (even if A is m-accretive), nor is it true for accretive operators that do not satisfy the range condition (even if E is Hilbert). In the setting of this theorem, $cl(R(A))$ is not convex in general, even if E is Hilbert. (It is convex if A is m-accretive.)

Theorem 2. *Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies $R(I+rA) \supset cl(D(A))$ for all $r>0$, S the semigroup generated by $-A$, and J_t the resolvent of A. If E is (UG) and E^* is (F), then for each x in $cl(D(A))$, $\lim_{t\to\infty} S(t)x/t = \lim_{t\to\infty} J_t x/t = -v$, where v is the point of least norm in $cl(R(A))$.*

Proof. Let $d = d(0, R(A)) = d(0, clco(R(A)))$ by Theorem 1. We always have $\limsup_{t\to\infty} |x - S(t)x|/t \leqslant d$ and $\lim_{t\to\infty} |J_t x/t| = d$. Since $(x - S(t)x)/t$ belongs to $clco(R(A))$, we also have $|(x - S(t)x)/t| \geqslant d$ for all t. Thus $\lim_{t\to\infty} |(x - S(t)x)/t| = d$. Since E^* is (F), every sequence $\{x_n\}$ in a convex subset K such that $\lim_{n\to\infty} |x_n| = d(0, K)$ converges. Hence the result.

If E is (UG), reflexive, and strictly convex, then $S(t)x/t$ and $J_t x/t$ converge weakly as $t \to \infty$. Several known results can now be improved. For example, [1, Corollary 4.3] is now seen to be true for all closed convex C.

Theorem 3. *Let C be a closed convex subset of a Banach space E, and let $T: C \to C$ be nonexpansive. Let the sequence $\{x_n : n = 0, 1, 2, \cdots\}$ be defined by $x_{n+1} = c_n T x_n + (1 - c_n) x_n$, where $x_0 \in C$ and $\{c_n\}$ is a real sequence such that $0 < c_n \leqslant 1$ and $a_n = \sum_{i=0}^{n} c_i \xrightarrow[n\to\infty]{} \infty$. If E is (UG) and E^* is (F), then the strong $\lim_{n\to\infty} x_{n+1}/a_n = -v$, where v is the point of least norm in $cl(R(I-T))$.*

Proof. We can apply Theorem 1 and the proof of Theorem 2 because $I - T$ is accretive and satisfies the range condition.

It follows that [1, Corollary 2.3] is true for all closed convex C. Kohlberg and Neyman [3] have established Theorem 3 for uniformly convex E in case $c_n = 1$ for all n. We have modified their idea to show that Theorem 1 is true if E is uniformly convex and smooth (equivalently, E is (G) and E^* is (UF)), and that Theorems 2 and 3 are valid if E is assumed to be uniformly convex. (See also [2] for other results of this type.)

It is expected that full details of these and other results (e.g. on infinite products of resolvents) will appear elsewhere.

References

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