20. *A Note on Asymptotic Strong Convergence of Nonlinear Contraction Semigroups*

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1. Let H be a real Hilbert space with inner product $(\ , \ )$, φ a proper l.s.c. (lower semicontinuous) convex functional on H into $(-\infty, \infty]$ such that the effective domain contains 0, and let $\partial\varphi$ be the subdifferential of φ. The operator $\partial\varphi$ generates a contraction semigroup, say $\{S(t)\}$, and $u(t)=S(t)x$ is an absolutely continuous solution of the initial-value problem

$$(1)\qquad \begin{cases} \dfrac{du}{dt} \in -\partial\varphi(u(t)) & \text{a.e.} \, t \in (0, \infty), \\ u(0)=x \in \overline{D(\varphi)}, \end{cases}$$

where $D(\varphi)=\{x\,;\,\varphi(x)<\infty\}$.

Recently R. Bruck [2] has treated the asymptotic strong convergence of solutions to the initial-value problem for (1) under the assumption that φ is even, i.e., $\varphi(x)=\varphi(-x)$ for $x \in D(\varphi)$. In this note we show that his approach also works for a more general case than that of even convex functionals.

Our result is stated as follows:

Theorem. *If there is a positive number α such that*

$$(2)\qquad \varphi(x)-\varphi(0) \geqq \alpha\{\varphi(-x)-\varphi(0)\} \qquad \textit{for } x \in D(\varphi),$$

then the solution $u(t)$ of the equation (1) *converges strongly as $t\to\infty$ to some minimum point of φ. That is, $s-\lim_{t\to\infty} u(t) \in F=\{x \in H : \varphi(x) =\inf \varphi\}$.*

A functional φ is even iff the inequality (2) holds for $\alpha=1$; hence our result extends Theorem 5 of Bruck [2, p. 23]. Although the proof of the above theorem is obtained by the method due to Bruck, condition (2) is considerably weaker than the assumption that φ is even.

2. **Proof of Theorem.** If $\alpha>1$, condition (2) implies that φ is constant. So, in what follows, we assume that $0<\alpha\leqq 1$. Moreover, we may assume

$$\varphi(0)=0,$$

since we have trivially $\partial\varphi=\partial(\varphi+\text{const.})$. Then, we have

$$(3)\qquad \varphi(x) \geqq \alpha\varphi(-x)=\alpha\varphi(-x)+(1-\alpha)\varphi(0) \geqq \varphi(-\alpha x)$$

for every $x \in D(\varphi)$.

The origin 0 is a minimum point of φ. In fact, for each $x \in D(\varphi)$, the inequality (3) implies

$$\varphi(x)=\frac{1}{1+\alpha}\varphi(x)+\frac{\alpha}{1+\alpha}\varphi(x)\geqq\frac{1}{1+\alpha}\varphi(-\alpha x)+\frac{\alpha}{1+\alpha}\varphi(x)$$
$$\geqq\varphi\left(\frac{-\alpha}{1+\alpha}x+\frac{\alpha}{1+\alpha}x\right)=\varphi(0).$$

From the property of the subdifferential it follows that, for the solution $u(t)$ of the equation (1), $\varphi(u(t))$ is decreasing in $(0, \infty)$. So, using the definition of the subdifferential and the inequality (3), we have

$$\begin{aligned}\varphi(u(t))&\geqq\varphi(u(t_0))\geqq\varphi(-\alpha u(t_0))\\&\geqq\varphi(u(t))+\left(-\frac{du}{dt}(t),\ -\alpha u(t_0)-u(t)\right)\end{aligned}\tag{4}$$

for a.e.t $\in [0, t_0]$.

Let $t_0>0$ be fixed. We define a functional $g: [0, t_0]\to(-\infty, \infty)$ by

$$g(t)=\frac{1+\alpha}{2}(\|u(t)\|^2-\|u(t_0)\|^2)-\frac{\alpha}{2}\|u(t)-u(t_0)\|^2.$$

The inequality (4) implies

$$\frac{dg}{dt}(t)=\left(\frac{du}{dt}(t), \alpha u(t_0)+u(t)\right)\leqq 0$$

for a.e.t $\in [0, t_0]$. Note that $g(t_0)=0$. Since $u(t)$ is absolutely continuous, so is $g(t)$. Therefore we have $g(t)\geqq 0$ if $0<t<t_0$, i.e.,

$$\frac{1+\alpha}{2}(\|u(t)\|^2-\|u(t_0)\|^2)\geqq\frac{\alpha}{2}\|u(t)-u(t_0)\|^2\tag{5}$$

if $0<t<t_0$. This implies that $\frac{1+\alpha}{2}\|u(t)\|^2$ is decreasing in t. Hence, again by (5), we get

$$\frac{\alpha}{2}\|u(t)-u(t_0)\|^2\to 0 \qquad \text{as } t\to\infty.$$

So, u(t) converges strongly to some point of H which is a minimum point of φ (cf. [2]).

References

[1] H. Brezis: Asymptotic behavior of some evolution systems. Nonlinear Evolution Equations (ed. by M. G. Crandall). Academic Press, New York, pp. 141–154 (1978).

[2] R. Bruck: Asymptotic convergence of nonlinear contraction semigroups in Hilbert space. J. Funct. Anal., **18**, 15–26 (1975).