13. *A Remark on the Inverse Theorem of Cauchy-Kowalevski**)

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1. Introduction. The purpose of this paper is to show the inverse theorem of Cauchy-Kowalevski. Consider the following Cauchy problem in a neighbourhood of the origin of C^{n+1}:

$$a(D_x, D_t)u(x,t)=f(x,t), \tag{1.1}$$

$$D_t^k u|_{t=0}=u_k(x), \qquad 0\leqq k<m, \tag{1.2}$$

where $(x,t)=(x_1,\cdots,x_n,t)\in C^{n+1}$ and $(D_x, D_t)=(\partial/\partial x_1,\cdots,\partial/\partial x_n,\partial/\partial t)$.

As well-known, Cauchy-Kowalevski's theorem says that if the operator $a(D_x, D_t)$ is Kowalevskian of order m with respect to D_t, that is,

$$a(D_x, D_t)=D_t^m+\sum_{j=1}^m a_j(D_x)D_t^{m-j}, \qquad \text{order } a_j(D_x)\leqq j, \tag{1.3}$$

then there exists a unique holomorphic solution $u(x,t)$ at the origin for any holomorphic function $f(x,t)$ at the origin and any holomorphic Cauchy data $\{u_k\}_{k=0}^{m-1}$ at the origin.

Our purpose is to show the converse. That is,

Theorem 1. *The Cauchy-Kowalevski theorem holds for the problem* (1.1)–(1.2) *if and only if* $a(D_x, D_t)$ *is Kowalevskian of order* m *with respect to* D_t.

Concerning this problem, some results were obtained in the case when $a(x,t;D_x,D_t)=D_t^m+\sum_{j=1}^m a_j(x,t;D_x)D_t^{m-j}$ (see Mizohata [6], [7], Miyake [2] and Kitagawa-Sadamatsu [1]). In the case of system of partial differential equations which is written in a normal form with respect to the time variable, the author [3] and Mizohata [5] obtained some results. Our interest here is to consider the problem without such restriction on the operator $a(D_x, D_t)$.

Our theorem will be proved in § 3, by means of Theorem 2 which concerns the Goursat problem. We have to mention that our theorem can not be extended to the case of variable coefficients by the same method as in this paper.

2. Holomorphic Goursat problem. Consider the following Goursat problem in $C_x^n\times C_t^1$:

$$D_x^\alpha D_t^m u(x,t)=\sum_{j=0}^m a_j(D_x)D_t^{m-j}u(x,t)+f(x,t), \tag{2.1}$$

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(2.2) $$u(x,t)-w(x,t)=O(x^{\alpha}t^{m}).$$

Let $0<d,\ \varepsilon\leqq 1,\ R,S>0,\ k\in N=\{0,1,2,\cdots\}$ and $\xi\in R_{+}^{n}$, $(R_{+}=(0,+\infty))$. Then the Banach space $A_{R,S,k}^{d,\varepsilon}(\xi)$ is defined by

(2.3) $$\|u\|_{k,\xi,R,S} =\inf\left\{C\,;\,|D_x^{\alpha}D_t^{m}u(0,0)|\leqq C\frac{(d|\alpha|+\varepsilon m+k)!}{R^{|\alpha|}S^{m}}\xi^{\alpha},\ {}^{\forall}\alpha,\ {}^{\forall}m\right\}<+\infty,$$

where $(d|\alpha|+\varepsilon m+k)!=\Gamma(d|\alpha|+\varepsilon m+k+1)$.

Especially we put $A_{R,S}^{d,\varepsilon}(\xi)=A_{R,S,0}^{d,\varepsilon}(\xi)$. Now $A^{\{d,\varepsilon\}}$ and $A^{(d,\varepsilon)}$ are defined by $A^{\{d,\varepsilon\}}=\bigcup_{R,S>0}A_{R,S}^{d,\varepsilon}(\xi)$ and $A^{(d,\varepsilon)}=\bigcap_{R,S>0}A_{R,S}^{d,\varepsilon}(\xi)$ respectively. Note that $A^{\{d,\varepsilon\}}$ and $A^{(d,\varepsilon)}$ do not depend on $\xi\in R_{+}^{n}$.

Now we study the Goursat problem (2.1)–(2.2) in the space $A^{\{d,\varepsilon\}}$ or $A^{(d,\varepsilon)}$. We assume the following conditions on $\{a_j(D_x)\}$:

(2.4) $$\varepsilon j\geqq d\{\text{order }a_j(D_x)-|\alpha|\},\qquad {}^{\forall}j\in\{0,1,\cdots,m\},$$

(2.5) $$\mathring{a}_0(\xi)\cdot\xi^{-\alpha}<1,\qquad {}^{\exists}\xi\in R_{+}^{n},$$

where $\mathring{a}_0(\xi)=\sum_{|\beta|=|\alpha|}|a_{\beta 0}|\,\xi^{\beta}$ and $a_0(D_x)=\sum_{|\beta|\leqq|\alpha|}a_{\beta 0}D_x^{\beta}$. Then we have

Theorem 2. *Assume the conditions* (2.4)–(2.5). *Then for any $f(x,t)$ and any $w(x,t)$ in $A^{\{d,\varepsilon\}}$ (resp. $A^{(d,\varepsilon)}$) there exists a unique solution $u(x,t)$ of* (2.1)–(2.2) *in $A^{\{d,\varepsilon\}}$ (resp. $A^{(d,\varepsilon)}$).*

First, we note that we may assume without loss of generality that $w(x,t)\equiv 0$ in (2.2). Then it is easy to see that the Goursat problem (2.1)–(2.2) is equivalent to the following integro-differential equation:

(2.6) $$U(x,t)=\sum_{j=0}^{m}a_j(D_x)D_x^{-\alpha}D_t^{-j}U(x,t)+f(x,t),$$

where $D_{x_j}^{-1}U(x,t)$ denotes the primitive of $U(x,t)$ with respect to x_j which vanishes at $x_j=0$.

Now the solution $u(x,t)$ of (2.1)–(2.2) is given by $u(x,t)=D_x^{-\alpha}D_t^{-m}U(x,t)$, where $U(x,t)$ is the solution of (2.6). For the proof of the unique existence of the solution of (2.6), we use the fixed point theorem in $A_{R,S,k}^{d,\varepsilon}(\xi)$ for a suitable choice of R,S,k and ξ.

Lemma. *We assume that $\alpha,\beta\in N^{n}$ and $j\in N$ satisfy $\varepsilon j\geqq d(|\beta|-|\alpha|)$. Then for any $U(x,t)\in A_{R,S,k}^{d,\varepsilon}(\xi)$ we have*

(2.7) $$\|D_x^{\beta}D_x^{-\alpha}D_t^{-j}U\|_{k,\xi,R,S}\leqq\frac{C}{k^{\delta}}\xi^{\beta-\alpha}\frac{S^{j}}{R^{|\beta|-|\alpha|}}\|U\|_{k,\xi,R,S},$$

where $\delta=\varepsilon j-d(|\beta|-|\alpha|)$, C is a positive constant independent of k,ξ,R and S and $C=1$ if $\varepsilon j=d(|\beta|-|\alpha|)$.

Proof. By a simple calculation we have

$$\|D_x^{\beta}D_x^{-\alpha}D_t^{-j}U\|_{k,\xi,R,S} \leqq\xi^{\beta-\alpha}\frac{S^{j}}{R^{|\beta|-|\alpha|}}\|U\|_{k,\xi,R,S}\sup_{\gamma,l}\frac{(d|\gamma|+\varepsilon l+k)!}{(d|\gamma|+\varepsilon l+k+\varepsilon j-d(|\beta|-|\alpha|))!}.$$

This implies (2.7) immediately. Q.E.D.

Proof of Theorem 2. In what follows, we fix $\xi\in R_{+}^{n}$ which satisfies the condition (2.5). We put

$$G=\sum_{j=0}^{m}a_j(D_x)D_x^{-\alpha}D_t^{-j}=\sum_{\beta}\sum_{j=0}^{m}a_{\beta j}D_x^{\beta}D_x^{-\alpha}D_t^{-j}.$$

First, consider the problem (2.6) in the space $A^{\{d,\varepsilon\}}$. Let $k=0$ in the above lemma. Then we have

$$\|G\|_{R,S} \leqq \mathring{a}_0(\xi)\cdot\xi^{-\alpha} + C \sum_{|\beta|<|\alpha|} |a_{\beta 0}|\cdot R^{|\alpha|-|\beta|} + C\sum_{\beta}\sum_{j=1}^{m} |a_{\beta j}| \frac{S^j}{R^{|\beta|-|\alpha|}} = \mathrm{I}+\mathrm{II}+\mathrm{III},$$

for some positive constant C. Here $\|G\|_{R,S}$ denotes the operator norm of G in the space $A^{d,\varepsilon}_{R,S}(\xi)$. Note that $\mathrm{I}<1$ by the condition (2.5). On the other hand, we have

$$\lim_{R\downarrow 0} \mathrm{II}=0 \quad \text{and} \quad \lim_{S\downarrow 0}\mathrm{III}=0 \qquad \text{for any fixed } R.$$

Hence G is a contraction map in $A^{d,\varepsilon}_{R,S}(\xi)$ for sufficiently small R and S.

Next, consider the problem in the space $A^{(d,\varepsilon)}$. In this case, we have

$$\|G\|_{R,S,k} \leqq \mathring{a}_0(\xi)\cdot\xi^{-\alpha} + C{\sum_{\beta}}' \sum_{j=1}^{m} |a_{\beta j}| \frac{S^j}{R^{|\beta|-|\alpha|}} + {\sum_{\beta}}''\sum_{j=0}^{m} \frac{C}{k^{\delta}} \frac{S^j}{R^{|\beta|-|\alpha|}} = \mathrm{I}+\mathrm{II}+\mathrm{III},$$

for some positive constants C and δ. Here $\|G\|_{R,S,k}$ denotes the operator norm of G in the space $A^{d,\varepsilon}_{R,S,k}(\xi)$, and $\sum'_{\beta}$ and $\sum''_{\beta}$ denote the summations over such β as $\varepsilon j = d(|\beta|-|\alpha|)$ and $\varepsilon j > d(|\beta|-|\alpha|)$ respectively.

Note that $\mathrm{I}<1$ by the condition (2.5). Let $S=a\cdot R^{\varepsilon/d}$, $(a>0)$. Then we have

$$\mathrm{II} = C{\sum_{\beta}}'\sum_{j=1}^{m} |a_{\beta j}|\, a^j \to 0 \qquad \text{as } a\downarrow 0.$$

On the other hand, we have $\lim_{k\uparrow\infty}\mathrm{III}=0$ for any fixed R. Hence, G is a contraction map in $A^{d,\varepsilon}_{R,aR^{\varepsilon/d},k}(\xi)$ for sufficiently small a and large k for any fixed R. This shows that the problem (2.6) is uniquely solvable in $\bigcap_{R>0} A^{d,\varepsilon}_{R,aR^{\varepsilon/d}}(\xi) = A^{(d,\varepsilon)}$. Q.E.D.

More general and detailed study of the holomorphic Goursat problem will be given in a forthcoming paper [4].

3. Proof of Theorem 1. It is obvious that in order that the Cauchy-Kowalevski theorem may hold for the problem (1.1)–(1.2), it is necessary that $a(D_x, D_t)$ is written in the form

$$a(D_x, D_t) = a_0(D_x, D_t)D_t^m + \sum_{j=1}^{m} a_j(D_x)D_t^{m-j}. \tag{3.1}$$

In order to complete the proof, it suffices to show $a_0(D_x, D_t) \equiv \text{Const.}\ (\neq 0)$ (see Mizohata [5]). We prove this by a principle of contradiction. Now we may assume that $a_0(D_x, D_t) \equiv a_0(D_x)$ and order $a_0(D_x) = l > 0$. Then by a suitable change of coordinates we may assume that $a_0(D_x) = D_{x_1}^l + \sum_{k=1}^{l} a_{0k}(D_{x'})D_{x_1}^{l-k}$ and order $a_{0k}(D_{x'}) \leqq k$, where $x=(x_1, x')$. Now let us consider the following Goursat problem:

$$a(D_x, D_t)u(x,t) = f(x,t), \qquad u(x,t) - w(x,t) = O(x_1^l t^m). \tag{3.2}$$

Choose d satisfying $d^{-1} \geqq \max\{1, (\text{order } a_j(D_x) - l)/j;\ j=1,2,\cdots,m\}$.

Then the above problem (3.2) is uniquely solvable in $A^{\{d,1\}}$ or $A^{(d,1)}$. Note that the condition (2.5) is satisfied by $\xi=(\xi_1, 1, \cdots, 1) \in \boldsymbol{R}_+^n$ with $\xi_1 \gg 1$. Hence, the arbitrariness of the Goursat data on the plane $x_1=0$ implies the non-uniqueness of the solution of the Cauchy problem (1.1)–(1.2). Q.E.D.

References

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